Day 17 - Generating Functions for Integer Partitions

Partition Identities

Recall:

- A sequence of positive integers $a_1 \geq a_2 \geq \ldots \geq a_m > 0$ is called an (integer) partition of $a$ provided that $a_1 + a_2 + \ldots + a_m = a$. Denote $p_k(n)$.
- The partition function $p(n)$ is the total number of integer partitions for $n$.
- $p_k(n)$ is the number of partitions of $n$ into $k$ parts.

Partition may be represented by Farey diagram.

- We write $|\lambda| = n$ to denote that there are $n$ boxes in the Farey diagram of $\lambda$.
- We write $\ell(\lambda) = k$ to denote that there are $k$ parts in $\lambda$.

| $\lambda$ | $\ell(\lambda)$ | $|\lambda|$ |
|----------|----------------|----------|
| $(5, 2, 2, 1)$ | 4 | 10 |
| $\lambda_1 = 5$ | $\lambda_2 = 2$ | $\lambda_3 = 2$ | $\lambda_4 = 1$ |

There are 10 cells in Farey diagram of $\lambda$.

For $\lambda$: $\# g$ parts of $\lambda = \# g$ rows in Farey diagram for $\lambda$.

Question: Given integers $2 \leq k \leq n$. How many partitions $\lambda$ of $n$ whose parts lengths are no longer than $k$ (i.e., $\lambda_i \leq k$) are there?

Answer: Let $p_k(n)$ denote the number of partitions of $n$ whose parts lengths are no longer than $k$.

Consider:

$$p_k(n) = \sum p_k(n) = \sum \left( \text{# of } \lambda \text{ with } \lambda_1 \leq k \right) \cdot \lambda^1_1 \lambda^2_2 \lambda^3_3 \ldots$$

Then:

$$p_k(n) = \prod \left( 1 \cdot \frac{1}{1-x^i} \right) \cdot \left( \frac{1}{1-x^k} \right) \cdot \ldots$$

$$p_k(n) = \sum_{\lambda: \lambda_i \leq k} \lambda_1 \lambda_2 \lambda_3 \ldots$$

A typical $\lambda$ that is counted by the sum above must have the form $\lambda = (k^m, (k-1)^m, \ldots, 1^m)$ (A has $m$ parts of length $i$).

$$\sum \lambda_1 \lambda_2 \lambda_3 \ldots = (1 + x + x^2 + \ldots) \cdot (1 + x^2 + x^4 + \ldots) \cdots (1 + x^k + x^{2k} + \ldots)$$

$$= \frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^3} \cdots \frac{1}{1-x^k}$$

$|\lambda| = \# g$ cells
\[
\sum_{n=0}^{\infty} \frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^3} \cdots \frac{1}{1-x^k}
\]

w/o the restriction that part length \( \leq k \).

\[
\sum_{n} x^{12} = \prod_{k=1}^{n} \frac{1}{1-x^k} : \text{Euler's identity.}
\]

\[p_k(n) = \text{number of partitions of } n \text{ w/ exactly } k \text{ parts.}\]

\[
\sum_{n=0}^{\infty} p_k(n) x^n = \sum_{\lambda : \ell(\lambda) = k} x^{\lambda_1}.
\]

We also know:

\[
P_k(x) = \sum_{\pi} \pi(x)^p = x^k \sum_{\pi} \pi(x)^p = \prod_{m=1}^{k} \frac{x^m}{1-x^m}.
\]

Take a pattern \( \lambda : \ell(\lambda) = k \).

\[|^{\lambda_1 = 5} \]

To construct a \( \mu \) w/ \( \ell(\mu) = k \).

- Start w/ a pattern \( \mu \) whose part lengths is at most \( k \).
- Attach a row of \( k \) cells on top of \( \mu \).
- Conjugate the result.

\[
\sum_{\lambda : \ell(\lambda) = k} x^{\lambda_1} = \prod_{m=1}^{k} \sum_{\mu : \ell(\mu) < k} x^{\mu_1} = \frac{x^k}{(1-x)(1-x^2) \cdots (1-x^k)}.
\]

Expand this, look at the coeff \( a \cdot x^k = p_k(n) \).
Let \( p_d(n) \) be the number of partitions of \( n \) into \( d \) parts of odd length. Let \( n_{od}(n) \) be the number of partitions of \( n \) into \( d \) parts with distinct lengths.

What are \( D(x) = \sum_{n \geq 0} p_d(n) x^n \) and \( D(x) = \sum_{n \geq 0} n_{od}(n) x^n \)?

\[
D(x) = \sum_{n \geq 0} p_d(n) x^n = \sum_{n \geq 0} x^{[\lambda]} = \sum_{n \geq 0} x^{[\lambda]}
\]

\[
= (1 + x) \cdot (1 + x^2) \cdot (1 + x^3) \cdot \ldots = \prod_{i=1}^{\infty} \frac{1}{1-x^i}
\]

\[
O(x) = \sum_{n \geq 0} n_{od}(n) x^n = \sum_{n \geq 0} x^{[\lambda]} = \sum_{n \geq 0} x^{[\lambda]}
\]

\[
= \frac{1}{1-x} \cdot \frac{1}{1-x^3} \cdot \frac{1}{1-x^5} \cdot \ldots = \prod_{i=1}^{\infty} \frac{1}{1-x^{2i-1}}
\]
\[ D(x) = \prod_{i=1}^{\infty} \left(1 + x^i\right) \quad \text{and} \quad \mathcal{O}(x) = \prod_{i=1}^{\infty} \frac{1}{1 - x^{2i-1}} \]

So

\[ \prod_{i=1}^{\infty} \frac{1 - x^i}{1 - x^{2i-1}} = \prod_{i=1}^{\infty} \frac{1}{1 - x^i} \]

So

\[ \sum_{n \geq 0} p_{\text{dist}}(n) x^n = D(x) = \mathcal{O}(x) = \sum_{n \geq 0} p_{\text{odd}}(n) x^n \]

\[ p_{\text{dist}}(n) = p_{\text{odd}}(n) \quad \text{for all } n \geq 0 \]
Example. Consider the sequence given by \( a_0 = 0 \) and
\[
    a_{n+1} = (n+1)a_n + 2(n+1)!
\]
for \( n \geq 0 \). Find the explicit formula for \( a_n \).