Definition 1. A stack is a last-in first-out linear sorting device that allows two operations push and pop.

procedure StackSort (σ = σ₁σ₂⋯σₙ: a permutation of length n)

1. Initialize an empty stack
2. for i := 1 to n
3. if stack is non-empty and σᵢ > first entry on stack then
4. pop the first entry on the stack to the output
5. else push σᵢ into the stack
6. while stack is non-empty pop the stack entries to the output

Example 1. Stack sort 3 2 1 4 5
Example 2. Stack sort 4 1 3 5 2

If \( s(p) \) = the identity permutation then we say \( p \) is \textbf{stack-sortable}.

Some observations: Let \( p \) be a permutation in one-line notation and let \( a < b \) be two entries of \( p \).

- If \( a \) precedes \( b \) in \( p \) then \( a \) also precedes \( b \) in \( s(p) \)

- If \( b \) precedes \( a \) in \( p \) and there is no element \( c \) located between \( a \) and \( b \) in \( p \) such that \( c > b > a \) then \( a \) precedes \( b \) in \( s(p) \)

- If \( b \) precedes \( a \) in \( p \) and there is an element \( c \) located between \( a \) and \( b \) in \( p \) such that \( c > b > a \) then \( b \) precedes \( a \) in \( s(p) \)
Definition 2. Given a sequence \( \tau = \tau_1 \cdots \tau_n \) of distinct positive integers, we define the reduction of \( \tau \), \( \text{red}(\tau) \), to be the permutation of \( S_n \) that results by replacing the \( i \)-th smallest element of \( \tau \) by \( i \) for each \( i \).

For example \( \text{red}(5 \ 3 \ 9 \ 6 \ 2) = 3 \ 2 \ 5 \ 4 \ 1 \).

Definition 3. Let \( \tau = \tau_1 \cdots \tau_j \in S_j \) and \( \sigma = \sigma_1 \cdots \sigma_n \in S_n \). Then

1. \( \tau \) occurs in \( \sigma \) if there exists \( 1 \leq i_1 < \cdots < i_j \leq n \) such that
   \[
   \text{red}(\sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_j}) = \tau.
   \]

2. \( \sigma \) avoids \( \tau \) is there is no occurrence of \( \tau \) in \( \sigma \). In this case, we let \( S_n(\tau) \) denote the set of permutations of \( S_n \) which avoid \( \tau \).

Definition 4. For a permutation \( \sigma = \sigma_1 \sigma_2 \cdots \sigma_n \), its permutation matrix is obtained by placing a mark in column \( i \) and row \( \sigma_i \) on an \( n \times n \) array, for each \( 1 \leq i \leq n \).

Theorem 1. A permutation \( p \) is stack-sortable if and only if it avoids the pattern 231.
Q: How many stack-sortable permutations of length $n$ are there?
Theorem 2.

The number of stacksortable permutations of length \( n = S_n(231) = c_n, \)
where
\[
c_n = \frac{1}{n+1} \binom{2n}{n}
\]
is the \( n \)-Catalan number.

Q: What about \( S_n(132) \)?
Q: What about \( S_n(132, 231) \)?

**Definition 5.** A permutation \( p \) is called two-stack sortable if \( s(s(p)) = \) the identity permutation.

**Theorem 3.** A permutation \( p \) is two-stack sortable if and only if it does not contain a 2341-pattern, and it does not contain a 3241-pattern except as a part of a 35241-pattern.

The number of two-stack sortable permutations of length \( n \) is

\[
\frac{2}{(n+1)(2n+1)} \binom{3n}{n}.
\]