Day 23 – Bases of Symmetric Function

Symmetric Functions

References:


Recall that a partition \( \lambda = (\lambda_1, \ldots, \lambda_r) \) is a finite sequence of weakly decreasing non-negative integers. We denote

- \( |\lambda| = \) sum of the integers in the partition \( \lambda \), and \( \text{(size)} \)
- \( \ell(\lambda) = \) number of parts in \( \lambda \).

If \( |\lambda| = n \) then we write \( \Lambda \vdash n \) to denote \( \lambda \) is a partition of \( n \).

**Definition 1.** A symmetric polynomial \( f \) in \( k \) variables \( x_1, x_2, \ldots, x_k \) is a polynomial that satisfies the property

\[
f(x_1, x_2, \ldots, x_k) = f(x_{\sigma_1}, x_{\sigma_2}, \ldots, x_{\sigma_k})
\]

for all permutation \( \sigma \in S_k \).

Let \( \Lambda_0(\chi_1, \chi_2, \ldots, \chi_k) \) denote the set of symmetric polynomials in the variable \( \chi_1, \chi_2, \ldots, \chi_k \) whose homogeneous degree is \( n \) and let \( \Lambda(\chi_1, \chi_2, \ldots, \chi_k) = \bigcup_{n \geq 0} \Lambda_n(\chi_1, \chi_2, \ldots, \chi_k) \) be the ring of symmetric functions in the variable \( \chi_1, \chi_2, \ldots, \chi_k \).

Each monomial w/\( n \) in the sym. poly \( f \) must have total degree \( n \).

* We write \( \Lambda, \Lambda_n \) for

\[
\Lambda (x_1, \ldots, x_k); \Lambda_n (x_1, \ldots, x_k)
\]

where the variables \( x_1, \ldots, x_k \) are unchanged in the problem.
Definition 2. For \( \lambda \vdash n \), the monomial symmetric function \( m_\lambda \) is the element in \( \Lambda_n \) given by the sum of all monomials where the exponents on the powers of \( x_i \) give a rearrangement of the parts of \( \lambda \).

Remark: \( m_\lambda \) is the sum of all monomials whose exponents can be rearranged to give the partition \( \lambda \).

\[
\begin{align*}
    m_{(3,1,1)}(x_1, x_2, x_3) &= x_1^3 x_2 x_3 + x_1 x_2^3 x_3 + x_1 x_3^3 x_2 \\
    m_{(2,2,1)}(x_1, x_2, x_3) &= x_1^2 x_2^2 + x_1^2 x_3^2 + x_2^2 x_1^2 + x_2^2 x_3^2 + x_3^2 x_2^2 + x_3^2 x_1^2
\end{align*}
\]

Let, all possible ways to muld. 1 copy of 1 variable, & 2 copies of another.

Lemma 1. The set \( \{m_\lambda : \lambda \vdash n\} \) is a basis for \( \Lambda_n \).

\[
\begin{align*}
\text{\{m_\lambda : \lambda \vdash n\} are indep:}  \\
\text{Take } \lambda, \mu \text{ as 2 different pttn's of n.}  \\
\Rightarrow m_\lambda \text{ and } m_\mu \text{ cannot have any monomial in common.}  \\
\Rightarrow \text{they're indep.}
\end{align*}
\]

\[
\begin{align*}
\text{\{m_\lambda : \lambda \vdash n\} spans } \Lambda_n  \\
\text{Take } p(x_1, \ldots, x_k) \text{ to be an element in } \Lambda_n  \\
\text{(every monomial/term in } p \text{ must have total degree } n.}  \\
\Rightarrow \text{we can write } p = \sum_{\lambda \vdash n} c_\lambda \cdot m_\lambda  \\
\text{for some const's } c_\lambda.
\end{align*}
\]

\[
\begin{align*}
\text{\{m_\lambda : \lambda \vdash n\} spans } \Lambda_n.  \\
\text{Dimension of } \Lambda_n = \# \text{ pttn of n = } \rho(n)
\end{align*}
\]
Fix $k$ variables $x_1, x_2, \ldots, x_k$.

**Definition 3.** The $n^{th}$ elementary symmetric function $e_n$ is defined through the generating function

$$E(z) = \sum_{n \geq 0} e_n z^n = \prod_{i \geq 1} (1 + x_i z) \rightarrow \text{expansion of this gives } e_n \text{'s}.$$ 

The $n^{th}$ **homogeneous symmetric function** $h_n$ is defined through the generating function

$$H(z) = \sum_{n \geq 0} h_n z^n = \prod_{i \geq 1} \frac{1}{1 - x_i z} \rightarrow \text{expand to get } h_n \text{'s}.$$

**In 3 variables** $x_1, x_2, x_3$:

$$E(z) = (1 + x_1 z)(1 + x_2 z)(1 + x_3 z)$$

$$= 1 + (x_1 + x_2 + x_3)z + (x_1 x_2 + x_1 x_3 + x_2 x_3)z^2 + (x_1 x_2 x_3)z^3$$

$$= \sum_{n=1}^3 \underbrace{e_n z^n}_{e_1 = 1, e_2 = 1}$$

$$H(z) = \frac{1}{1 - x_1 z} \cdot \frac{1}{1 - x_2 z} \cdot \frac{1}{1 - x_3 z}$$

$$= \left(1 + x_1 z + x_2^2 z^2 + \cdots \right) \left(1 + x_2 z + x_3^2 z^2 + \cdots \right)$$

$$= \left(1 + x_3 z + x_2^2 z^2 + \cdots \right)$$

$$= \sum_{n=1}^3 \underbrace{h_n z^n}_{h_1 = 1, h_2 = \text{other terms}}$$

**Remark:**

- $e_n$ is the sum of all square-free monomials of degree $n$.
- $h_n$ contains all possible degree $n$ monomials, each with a coefficient 1.
- $e_n = m_{(n)}$; $h_n = \sum_{\lambda \vdash n} m_{\lambda}$.
Definition 4. The $n^{th}$ power symmetric function $p_n$ in the variables $x_1, x_2, \ldots, x_k$ is defined as

$$p_n = x_1^n + x_2^n + \cdots + x_k^n.$$ 

Remark: $p_n = m(n).$

If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a partition then we define

$$e_\lambda = \prod_{i=1}^k e_{n_i}, \quad h_\lambda = \prod_{i=1}^k h_{n_i}, \quad p_\lambda = \prod_{i=1}^k p_{n_i},$$

$$\lambda = (5, 3, 1, 1)$$

$$e_\lambda = e_5 \cdot e_3 \cdot e_1 \cdot e_1$$

Fact: $\{e_\lambda : \lambda \vdash n\}, \{h_\lambda : \lambda \vdash n\}, \{p_\lambda : \lambda \vdash n\}$ are bases for $\Lambda_n.$

Definition 5. A Young tableau is a filling of the cells of a Ferrer diagram with positive integers. The tableau is called column strict if the integers inside satisfy two conditions:

1. The integers are strictly increase when reading from top to bottom within columns.

2. The integers are weakly increase when reading left to right within rows.

Ex: $\lambda = (4, 2, 1); \quad \text{alphabets} = \{1, 1, 1, 2, 2, 3, 3, 4\}$

\[
\begin{array}{c|c|c|c}
1 & 1 & 1 & 2 \\
2 & 4 & & \\
3 & & & \\
\end{array}
\quad
\begin{array}{c|c|c|c}
1 & 1 & 1 & 2 \\
2 & 3 & & \\
4 & & & \\
\end{array}
\quad
\begin{array}{c|c|c|c}
1 & 1 & 1 & 4 \\
2 & 2 & & \\
3 & & & \\
\end{array}
\quad
\begin{array}{c|c|c|c}
1 & 1 & 1 & 3 \\
2 & & & \\
4 & & & \\
\end{array}
\]

Let $CS_\lambda$ denote the set of all tableaux of shape $\lambda.$ Let $T \in CS_\lambda$ then the weight of $T$ is defined by $w(T) = \prod_{c \in T} x_c.$
\(e_n\) can be expressed as a sum of column strict tableaux of shape \(1^n\).

\[ e_n = \sum_{T \in S_{1^n}} w(T). \]

\(h_n\) can be expressed as a sum of column strict tableaux of shape \(n\).

\[ h_n = \sum_{T \in S_n} w(T). \]

Variables are \(x_1, x_2, \ldots, x_k\).

**Definition 6.** Let \(\lambda \vdash n\), the **Schur symmetric function** \(s_\lambda\) is defined as

\[ s_\lambda = \sum_{T \in S_\lambda} w(T). \]

\(\text{Ex: } \lambda = (2, 1)\)

\[ x_1 x_2 x_1 x_3 \]

- Look at all possible ways to fill the Ferrer diagram for \(\lambda\) using the alphabet \(\{1, 2, \ldots, k\}^2\).
- Number can be repeated.

\[
2 x_1 x_2 x_3 + x_1^2 x_2 + x_1 x_2^2 + 5 x_1 x_2 x_3 + x_2 x_3^2 + x_2 x_3 = S_{(2,1)}.
\]
Relationship between the bases

**Theorem 2.** For all $n \geq 1$, $\sum_{i=0}^{n} (-1)^i e_i h_{n-i} = 0.$
A second proof of this theorem
Theorem 3. For all $k, n \geq 1$, \( \sum_{i=0}^{n-1} (-1)^i e_i s_{i+1, n-1} = (-1)^{n-1} e_{n+k}. \)
Theorem 4. For all $k \geq 1$, \[
\sum_{n \geq 1} s_{(1^{k}, n)} z^{n+k} = \frac{1}{E(-z)} \cdot \sum_{n \geq k+1} (-1)^{n-k-1} e_{n} z^{n}.
\]