Instructions:

Homework should be done in groups of one to three people. You are free to change group members at any time throughout the quarter. Problems should be solved together, not divided up between partners. In your submission, you must indicate the name of each partner within your group. Each member of your group must submit his/her own work through Gradescope. Submissions must be received by 10:00pm on the due date, and there is no exception to this rule.

You will be able to look at your scanned work before submitting it. Please ensure that your submission is legible (neatly written and not too faint) or your homework may not be graded.

Students should consult their textbook, class notes, lecture slides, instructor, and TA’s when they need help with homework. Students should not look for answers to homework problems in other texts or sources, including the Internet. You may ask questions about the homework in office hours and on Piazza. However, you should only use Piazza to ask for clarifications on the homework problems. Publicly posting any part of your work on Piazza will be considered cheating, and the author of the post will receive a zero on the entire assignment!

Your assignments in this class will be evaluated not only on the correctness of your answers, but on your ability to present your ideas clearly and logically. You should always explain how you arrived at your conclusions and justify your answers with mathematically sound reasoning. Whether you use formal proof techniques or write a more informal argument for why something is true, your answers should always be well-supported. Your goal should be to convince the reader that your results and methods are sound.

Required Reading: Bóna Chapters 1, 2, and 3.

Key Concepts: Proof by induction and strong inductions, Pigeon-Hole principle, elementary counting problems.
1. Use mathematical induction to prove that the following equalities hold for all positive integers $n$.

(a.) $1 + 3 + 5 + \cdots + (2n - 1) = n^2$

(b.) $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$

2. Let $a_1 = 5$ and let $a_{n+1} = a_n^2$. Prove that the last $n$ digits of $a_n$ are the same as the last $n$ digits of $a_{n+1}$.

3. Prove that among any set of 51 positive integers less than 100, there is a pair whose sum is 100.

4. Let $n \geq 2$. We select a set $S$ of $n+1$ different integers from the set $[2n] = \{1, 2, \ldots, 2n\}$.

(a.) Prove that there exist two elements in $S$ with the greatest common divisor equal to 1.

(b.) Is it true that there will always exist two distinct elements $i, j \in S$ such that $j = 2i$? Prove or give a counter example.

(c.) Is it true that there will always exist two distinct elements $i, j \in S$ such that $j = ki$ for some integer $k \geq 2$? Prove or give a counter example.

5. Use the Pigeon-Hole Principle to prove that there exists a positive integer $n$ such that $44^n - 1$ is divisible by 7.

6. (a.) How many rearrangements are there of the word TENNESSEE?

(b.) How many of the rearrangements in part (a.) do not have the two S’s next to each other?

(c.) Let $k \leq n$. How many ways are there to place $k$ rooks on an $n \times n$ chessboard in such a way that no two rooks attack one another?

(d.) How many four-digit positive integers are there in which all the digits are different?

(e.) In how many ways can the elements of $[n] = \{1, 2, \ldots, n\}$ be permuted such that the sum of any two consecutive elements of the permutation is odd?

(f.) A TV show has 24 episodes. Due to the lack of time, you can only watch 5 of them. However, in order to get a gist of what the show is about, you do not want to watch any two consecutive episodes. How many selections of 5 episodes are there with no two consecutive episodes chosen? Assuming that the order of selection is irrelevant.

7. (Optional) Given a circle with diameter 5. We randomly choose 10 points on the circumference of this circle. Prove that there exist two points at a distance less than 2 from each other.