Instructions:

Homework should be done in groups of one to three people. You are free to change group members at any time throughout the quarter. Problems should be solved together, not divided up between partners. In your submission, you must indicate the name of each partner within your group. Each member of your group must submit his/her own work through Gradescope. Submissions must be received by 10:00pm on the due date, and there is no exception to this rule.

You will be able to look at your scanned work before submitting it. Please ensure that your submission is legible (neatly written and not too faint) or your homework may not be graded.

Students should consult their textbook, class notes, lecture slides, instructor, and TA’s when they need help with homework. Students should not look for answers to homework problems in other texts or sources, including the Internet. You may ask questions about the homework in office hours and on Piazza. However, you should only use Piazza to ask for clarifications on the homework problems. Publicly posting any part of your work on Piazza will be considered cheating, and the author of the post will receive a zero on the entire assignment!

Your assignments in this class will be evaluated not only on the correctness of your answers, but on your ability to present your ideas clearly and logically. You should always explain how you arrived at your conclusions and justify your answers with mathematically sound reasoning. Whether you use formal proof techniques or write a more informal argument for why something is true, your answers should always be well-supported. Your goal should be to convince the reader that your results and methods are sound.

Required Reading: Bóna Chapters 5.

Key Concepts: Compositions. Set partitions. Integer partitions. Bell numbers and Stirling numbers of the second kind.
1. **(Optional - Easy)** Build the table for the Stirling number of the second kind $S(n, k)$ for $1 \leq k \leq n \leq 6$, and use your result to find the first six Bell numbers $B(n)$ for $1 \leq n \leq 6$.

2. **(Optional - Easy)** List all integer partitions of 6 and decide which pair of partitions are conjugate of one another.

3. Give the closed formula for the Stirling numbers of the second kind using two methods:

   (1.) Derive directly from the recursion for the $S(n, k)$ and
   (2.) Through a counting argument.

   (a.) $S(n, 2)$ for $n \geq 2$.
   (b.) $S(n, n - 2)$ for $n \geq 2$.
   (c.) $S(n, n - 3)$ for $n \geq 3$.

4. Let $n$ be a positive integer and let $ST_n$ be the “staircase” board whose columns are of heights $0, 1, 2, \ldots, n - 1$ reading from left to right. Recall that in chess, a rook can attack all the squares on the same row and column as the cell it occupies.

   The figure below depicts the board $ST_7$ with one rook (denoted by the $X$ symbol). The red cells are all the cells that this rook is attacking. So if we want to place an additional rook piece on this board in a way that no rook can attack each other, we must avoid these red cells. We call these rook non-attacking.

   ![Staircase Board with Rook](image)

   (a.) **(Optional)** Construct a bijection to show that the number of ways to place $k$ non-attacking rooks on the board $ST_n$ equals to the number of set partitions of $[n]$ with $n - k$ blocks.

   (b.) The result in part (a) shows that the Stirling number of the second kind $S(n, k)$ is given by the number of ways to place $n - k$ non-attacking rooks on the staircase board $ST_n$. Use this interpretation to give a combinatorial proof for the recursion of $S(n, k)$ given by:

   $$ S(n, k) = S(n - 1, k - 1) + k \cdot S(n - 1, k), \text{ for all } 1 \leq k \leq n. $$
(c.) Let $x$ be a positive integer. We shall construct the augmented board $B_{x,n}$ from $ST_n$ by attaching $x$ rows of length $n$ below the board $ST_n$. Thus, the columns of $B_{x,n}$ are now of height $x, x + 1, \ldots, x + n - 1$. Count the number of ways we can place $n$ non-attacking rooks on the augmented board $B_{x,n}$ in two different ways to obtain the identity

$$x^n = \sum_{k=0}^{n} S(n, k) \cdot (x)_k.$$ 

5. This problem is about proving several identities for the Bell numbers $B(n)$.

(a.) Prove that for all non-negative integer $n$,

$$B(n + 1) = \sum_{i=0}^{n} \left( \begin{array}{c} n \\ i \end{array} \right) B(i).$$

(b.) Let $P(n)$ be the number of partitions of $[n]$ with no singleton block. Construct a bijection to show that $B(n) = P(n) + P(n + 1)$.

(c.) Let $B_k(n)$ be the number of partitions of $[n]$ such that if $i$ and $j$ are in the same block, then $|i - j| > k$. Prove that $B_k(n) = B(n - k)$ for all $n \geq k$.

(d.) (Optional) Can you give a combinatorial proof to the result in part (a) using the rook model in the previous problem?

6. Show that for all $k \geq 0$,

$$\sum_{n \geq k} S(n, k) t^n = \frac{t^k}{(1 - t)(1 - 2t) \cdots (1 - kt)}.$$ 

7. Let $F_k(x) = \sum_{n \geq k} S(n, k) \frac{x^n}{n!}$.

(a.) Show that

$$\frac{d}{dx} (F_k(x)) = F_{k-1}(x) + kF_k(x).$$

(b.) Use the result in part (a) and a proof by induction to show that

$$F_k(x) = \frac{(e^x - 1)^k}{k!}.$$