Instructions:

Homework should be done in groups of one to three people. You are free to change group members at any time throughout the quarter. Problems should be solved together, not divided up between partners. In your submission, you must indicate the name of each partner within your group. Each member of your group must submit his/her own work through Gradescope. Submissions must be received by 10:00pm on the due date, and there is no exception to this rule.

You will be able to look at your scanned work before submitting it. Please ensure that your submission is legible (neatly written and not too faint) or your homework may not be graded.

Students should consult their textbook, class notes, lecture slides, instructor, and TA’s when they need help with homework. Students should not look for answers to homework problems in other texts or sources, including the Internet. You may ask questions about the homework in office hours and on Piazza. However, you should only use Piazza to ask for clarifications on the homework problems. Publicly posting any part of your work on Piazza will be considered cheating, and the author of the post will receive a zero on the entire assignment!

Your assignments in this class will be evaluated not only on the correctness of your answers, but on your ability to present your ideas clearly and logically. You should always explain how you arrived at your conclusions and justify your answers with mathematically sound reasoning. Whether you use formal proof techniques or write a more informal argument for why something is true, your answers should always be well-supported. Your goal should be to convince the reader that your results and methods are sound.

Required Reading: Bóna Chapters 8 and 14.

1. A Protoss Executor has $n$ probes. He distributes the probes into some number of bases. For each base, he assigns three probes to mine gas while the rest are on the mineral line. Let $p_n$ denote the number of ways the Executor can proceed. Find the exponential generating function $P(x) = \sum_{n \geq 0} p_n \frac{x^n}{n!}$

You don’t need to construct additional pylons for this problem.

2. Let $n \geq 1$ be an integer. Suppose that we sit $2n$ people around a circular table. Let $h_n$ denote the number of ways that everyone can be simultaneously shaking hands with another person at the table in such a way that none of the arms cross each other. Prove that $h_n = C_n$ where $C_n$ is the $n$-th Catalan number defined in HW5.

3. Let $p = p_1 p_2 \cdots p_n \in S_n$ be a permutation in one-line notation. The descent set of $p$ is defined as

$$\text{Des}(p) = \{ i : 1 \leq i \leq n - 1 \text{ and } p_i > p_{i+1} \}.$$ 

We let $\text{des}(p) = |\text{Des}(p)|$ denote the number of descents in $p$.

We say that $p_i$ is a left-to-right minimum of $p$ if $p_k > p_i$ for all $k < i$. For example, the left-to-right minima of $\sigma = 938471625$ are 9, 3 and 1.

Let $p = p_1 p_2 \cdots p_n$ be a 132-avoiding permutation. Prove that for any $2 \leq i \leq n$, the entry $p_i$ is a left-to-right minimum of $p$ if and only if $i - 1$ is a descent of $p$.

4. Let $A$ be a set of permutation patterns. We let $S_n(A)$ denote the number of permutations of length $n$ that avoid all patterns in $A$.

(a.) Find a recurrence relation for the sequence $a_n = S_n(132, 321)$, and use it to prove that $a_n = 1 + \binom{n}{2}$.

(b.) Find a recurrence relation for the sequence $b_n = S_n(132, 4321)$, and use it to prove that $b_n = 2 \binom{n}{4} + \binom{n + 1}{3} + 1$.

5. A permutation $p$ in one-line notation is called an alternating permutation provided that $p_1 < p_2 > p_3 < p_4 > p_5 < \cdots$. Let $a_0 = 1$ and let $a_n$ denote the number of alternating permutations of length $n \geq 1$.

(a.) Let $O(x) = \sum_{n \geq 0} a_{2n+1} \frac{x^{2n+1}}{(2n+1)!}$. Show that $O(x)$ satisfies the recursion

$$\frac{d}{dx}[O(x)] = 1 + [O(x)]^2.$$ 

Use this result to find the closed formula for $O(x)$. 

(b.) Let $E(x) = \sum_{n \geq 0} a_{2n} \frac{x^{2n}}{(2n)!}$. Show that $E(x)$ satisfies the recursion
\[
\frac{d}{dx}[E(x)] = E(x)O(x).
\]
Use this result to find the closed formula for $E(x)$.

(c.) Find the exponential generating function $A(x) = \sum_{n \geq 0} a_n \frac{x^n}{n!}$. 