Math 261A Assignment 1: Remmel
Due Friday, October 23, 2009.

(1) A directed graph $G = (V, E)$ has a Hamiltonian path if there is a pair of vertices $s, t \in V$ and a directed path from s to t that goes through every vertex in V exactly once. Prove that every tournament on n vertices with $n \geq 2$ has at least one Hamiltonian path. (This does not require a probabilistic argument).

(2) Suppose that $X, Y : [a, b] \to R$ are continuous random variables.
(i) Show that for all constants c and d, $E[aX + bY] = aE[X] + bE[Y]$.
(ii) Show that if X and Y are independent, then $E[XY] = E[X]E[Y]$.

(3) A set 0-1 strings on length n is (n, k)-universal if, for any subset of k coordinates $S = \{i_1, \ldots, i_k\}$, the projection

$$A \restriction_S = \{(a_{i_1}, \ldots, a_{i_k}) : (a_1, \ldots, a_n) \in A\}$$

of A onto the coordinates in S contains all 2^k possibilities. Show that if $\binom{n}{k} 2^k (1 - 2^{-k})^r < 1$, then there is (n, k)-universal set of size r.

(4) Suppose that $n \geq 4$ and H is an n-uniform hypergraph with at most $\frac{4^{n-1}}{3^n}$ edges. Prove that there is a coloring of the vertices of H by four colors so that in every edge all four colors are represented. (This problem 2 on page 11 of Alon-Spencer.)

(5) Suppose that $n \geq 2$ and H is an n-uniform hypergraph with at most 4^{n-1} edges. Prove that there is a coloring of the vertices of H by four colors so that no edge is monochromatic. (This is problem 1 on page 23 of Alon-Spencer.)

(6) Let H be a hypergraph in which any two edges have at least two vertices in common. Prove that the family of edges of H has property B. (See page 7 of Alon-Spencer.)

(7) Let $\{(A_i, B_i) : i = 1, \ldots, h\}$ be a family of pairs of subsets of the set of integers such that $|A_i| = k$ for all i, $|B_i| = \ell$ for all i, $A_i \cap B_i = \emptyset$ for all i, and $(A_i \cap B_j) \cup (A_j \cap B_i) \neq \emptyset$ for all $i \neq j$. Prove that $h \leq \frac{(k+\ell)^{k+\ell}}{k^k \ell^\ell}$. (This is problem 7 on page 12 of Alon-Spencer.)