Rook Theory Notes

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1 Rook and Hit numbers and File and Fit numbers

1.1 Preliminaries

Throughout this book, we abbreviate left-hand-side and right-hand-side by LHS and RHS, respectively. We let $[n] = \{1, \ldots, n\}$. We let $\mathbb{N} = \{0, 1, 2, \ldots\}$ denote the natural numbers, $\mathbb{P} = \{1, 2, \ldots\}$ denote the positive integers, $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$ denote the integers, $\mathbb{C}$ denote the complex numbers, and $\mathbb{R}$ denote the real numbers. For any statement A, we let $\chi(A) = 1$ is A is true and $\chi(A) = 0$ if A is false.

Given a ring R, we let $R[x]$ denote the ring of polynomials in x, i.e. $R[x]$ consists of all polynomials $a_0 + a_1 x + \cdots + a_n x^n$ where $a_i \in R$ for all i and the operations on $R[x]$ is the usual addition and multiplication of polynomials. Let $R[[x]]$ denote the ring of formal power series in x, i.e. $R[[x]]$ consists of all formal power series $f = \sum_{n \geq 0} f_n x^n$ where $f_n \in R$. If $f = \sum_{n \geq 0} f_n x^n$ and $g = \sum_{n \geq 0} g_n x^n$ are elements of $R[[x]]$ and $c \in R$, then

$$\begin{aligned} cf &= \sum_{n \geq 0} c f_n x^n, \\ f + g &= \sum_{n \geq 0} (f_n + g_n) x^n \text{ and} \\ fg &= \sum_{n \geq 0} (\sum_{k=0}^{n} f_k g_{n-k}) x^n. \end{aligned}$$

If $f = \sum_{n \geq 0} f_n x^n$ is an element of $R[[x]]$, we shall let $f|_{x^k} = f_k$ denote the operation of taking the coefficient of x^k of n. If $\vec{a} = a_0, a_1, \ldots$ is a sequence of elements from R, then the *ordinary generating function of* $\vec{a}$ is just the formal power series $OGF_{\vec{a}}(x) = \sum_{n \geq 0} a_n x^n$ and if $\mathbb{Q} \subset R$, the *exponential generating function of* $\vec{a}$ is just the formal power series $EGF_{\vec{a}}(x) = \sum_{n \geq 0} \frac{a_n}{n!} x^n$. We shall write $\frac{1}{1-x}$ for the formal power series $\sum_{n \geq 0} x^n$. Finally, we need to define when a

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sequence of formal power series $\{g_m\}_{m\geq 0}$ converges to a formal power series g in $R[[x]]$. If $g_m = \sum_{n\geq 0} g_{n,m}x^n$ for $m \geq 0$ and $g = \sum_{n\geq 0} g_n x^n$, then we write

$$\lim_{m\to\infty} g_m = g \tag{1}$$

if and only if for all $k \geq 0$, there is an number $N_k > 0$ (depending on k) such that for all $m \geq N_k$, $g_{m,k} = g_k$.

1.2 Introduction to Rook Theory

The theory of rook polynomials was introduced by Kaplansky and Riordan [**?**], and developed further by Riordan [48]. We refer the reader to Stanley [50, Chap. 2] for a nice exposition of some of the basics of rook polynomials and permutations with forbidden positions.

A *board* is a subset of an $\mathbb{P}\times\mathbb{P}$. We label the rows of $\mathbb{P}\times\mathbb{P}$ from bottom to top with $1, 2, 3, \ldots$, and the columns of $\mathbb{P}\times\mathbb{P}$ from left to right with $1, 2, 3, \ldots$, and let (i, j) denote the square in the i-th row and j-th column. Given $b_1, \ldots, b_n \in \mathbb{N}$, we let $F(b_1, \ldots, b_n)$ denote the board consisting of all the cells $\{(i, j) : 1 \leq i \leq n \;\&\; 1 \leq j \leq b_i\}$. If a board B is of the form $B = F(b_1, \ldots, b_n)$, then we say that B is *skyline board* and if, in addition, $b_1 \leq b_2 \leq \cdots \leq b_n$, then we say that B is a *Ferrers board*. For example, Figure 1 pictures the skyline board $F(2, 1, 0, 3)$ on the left and the Ferrers board $F(1, 2, 2, 3)$ on the right where we have shaded the cells belonging to the board.

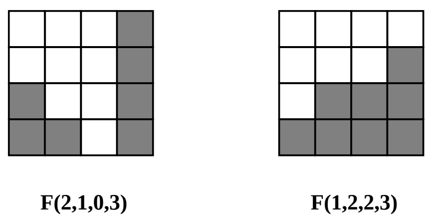

Figure 1: A skyline board and a Ferrers boards.

Next we introduce the two of the key objects of study of rook theory which are the rook numbers and file numbers of a board B.

Definition 1. *Given a board $B \subseteq [n] \times [n]$, we let*

1. *$\mathcal{N}_k(B)$ denote the set of all placements of k rooks in B such that no two rooks lie in the same row or column and*

2. *$\mathcal{C}_k(B)$ denote the set of placements of k rooks in B such that there is at most one rook in each column.*

We shall refer to an element $P \in \mathcal{N}_k(B)$ as a *placement of k non-attacking rooks in B* and element $Q \in \mathcal{C}_k$ as a *file placement of k rooks in B*. For $k = 1, \ldots, n$, we let $r_k(B) = |\mathcal{N}_k(B)|$ and $f_k(B) = |\mathcal{C}_k(B)|$. By convention, we set $r_0(B) = f_0(B) = 1$. We refer to $r_k(B)$ as the **k-th rook number of B** and $f_k(B)$ as the **k-th file number of B**.

Another important set of numbers associated with a board B are the hit numbers and the fit numbers. Let S_n denote the symmetric group, i.e. S_n is the set of all 1:1 functions $\sigma : [n] \to [n]$

under composition. Elements of S_n can be thought of as permutations of $1, 2, \ldots, n$. That is, if $\sigma \in S_n$ and $\sigma(i) = \sigma_i$ for $i = 1, \ldots, n$, then we shall write σ in two-line notation as

$$\begin{matrix} 1 & 2 & 3 & \cdots & n \\ \sigma_1 & \sigma_2 & \sigma_3 & \cdots & \sigma_n \end{matrix}.$$

We shall also write σ in one-line notation as $\sigma = \sigma_1 \ldots \sigma_n$. We shall also use cycle notation for permutations. For example, if $\sigma = 3\ 2\ 4\ 5\ 1\ 7\ 6\ 8$, then in cycle notation, $\sigma = (1, 3, 4, 5)(2)(6, 7)(8)$. Given a permutation $\sigma = \sigma_1 \ldots \sigma_n \in S_n$, we identify each $\sigma \in S_n$ with the rook placement $\{(\sigma_i, i) : i = 1, \ldots, n\}$ on $[n] \times [n]$. We let $\mathbb{F}_n$ denote the set of all functions $f : [n] \rightarrow [n]$. We will identify $f \in \mathbb{F}_n$ with the rook placement $\{(f(i), i) : i = 1, \ldots, n\}$ on $[n] \times [n]$. For example, if $\sigma = 2\ 3\ 1\ 5\ 4 \in S_n$ and f is the function given by $f(1) = 3$, $f(2) = 1$, $f(3) = 3$, $f(4) = 1$, and $f(5) = 4$, then the rook placement associated with σ is given on the left in Figure 2 and the file placement associated with f is given on the right in Figure 2. We let

$$\begin{aligned} H_{k,n}(B) &= |\{\sigma \in S_n : |\sigma \cap B| = k\}| \text{ and} \\ F_{k,n}(B) &= |\{f \in \mathbb{F}_n : |f \cap B| = k\}|. \end{aligned}$$

We shall refer to $H_{k,n}(B)$ as the **k-th hit number of B relative to** $[n] \times [n]$ and $F_{k,n}(B)$ as the **k-th fit number of B relative to** $[n] \times [n]$.

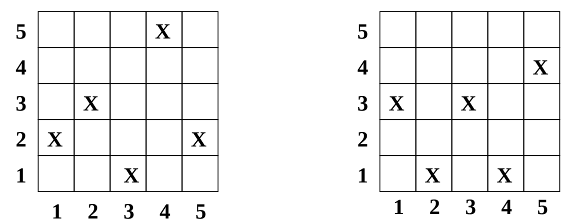

Figure 2: Rook placements associated with permutations and functions.

Kaplansky and Riordan [38] proved the following fundamental relationship between the rook numbers and the hit numbers of a board $B \subseteq [n] \times [n]$.

Theorem 1. *For any board $B \subseteq [n] \times [n]$,*

$$\sum_{k=0}^{n} H_{k,n}(B) x^k = \sum_{k=0}^{n} r_k(B)(n-k)!(x-1)^k. \tag{2}$$

Proof. Replacing x by $x + 1$ in equation (2), we see that it is enough to prove

$$\sum_{k=0}^{n} H_{k,n}(B)(x+1)^k = \sum_{k=0}^{n} r_k(B)(n-k)! x^k. \tag{3}$$

To interpret the LHS of (2), consider the set of objects O obtained by first picking a a rook placement on $[n] \times [n]$ associated with some permutation $\sigma \in S_n$ and then for each rook r which is in $\sigma \cap B$, we either circle the rook or not. For each such object O constructed in this way, we define the weight of O as $x^{circ(O)}$ where $circ(O)$ is the number of circled rooks in O. If $\mathcal{O}_B$ is the set of objects constructed in this way, it is easy to see that

$$\sum_{O \in \mathcal{O}_B} x^{circ(O)} = \sum_{k=0}^{n} H_{k,n}(B)(x+1)^k. \tag{4}$$

However, we can also count the LHS of (4) by first picking the number k of circled rooks, then picking the k circled rooks as a placement P of k non-attacking rooks on B which can be done in $r_k(B)$ ways, and finally extending P to a placement corresponding to a permutation $\sigma \in S_n$ by adding $n-k$ non-circled rooks which can done in $(n-k)!$ ways. Thus

$$\sum_{O \in \mathcal{O}_B} x^{circ(O)} = \sum_{k=0}^{n} r_k(B)(n-k)!x^k \tag{5}$$

which proves (3). $\square$

We can prove the fundamental relation between the file and fit numbers of a board $B \subseteq [n] \times [n]$ in the same way.

Theorem 2. *For any board $B \subseteq [n] \times [n]$,*

$$\sum_{k=0}^{n} F_{k,n}(B)x^k = \sum_{k=0}^{n} f_k(B)n^{n-k}(x-1)^k. \tag{6}$$

Proof. Again, replacing x by $x+1$ in equation (6), we see that it is enough to prove

$$\sum_{k=0}^{n} F_{k,n}(B)(x+1)^k = \sum_{k=0}^{n} f_k(B)n^{n-k}x^k. \tag{7}$$

To interpret the LHS of (7), consider the set of objects Q obtained by first picking a file placement on $[n] \times [n]$ associated with some function $f \in \mathbb{F}_n$ and then for each rook r which is in $f \cap B$, we either circle the rook or not. For each such object Q constructed in this way, we define the weight of Q as $x^{circ(Q)}$ where $circ(Q)$ is the number of circled rooks in Q. If $\mathcal{Q}_B$ is the set of objects constructed in this way, it is easy to see that

$$\sum_{Q \in \mathcal{Q}_B} x^{circ(O)} = \sum_{k=0}^{n} F_{k,n}(B)(x+1)^k. \tag{8}$$

However, we can also count the LHS of (8) by first picking the number k of circled rooks, then picking the k circled rooks as a file placement P of k rooks on B which can be done in $f_k(B)$ ways, and finally extending P to a placement corresponding to a function $f \in \mathbb{F}_n$ by adding $n-k$ non-circled rooks which can done in n^{n-k} ways. Thus

$$\sum_{Q \in \mathcal{Q}_B} x^{circ(O)} = \sum_{k=0}^{n} f_k(B)n^{n-k}x^k \tag{9}$$

which proves (7). $\square$

Next we shall prove two factorization theorems for polynomials associated with the rook numbers associated to a Ferrer's board and the file numbers associated with a skyline board. First we present a result of Goldman, Joichi, and White [21] for polynomials associated with rook numbers of Ferrers boards. We define the falling factorial polynomial $(x)\downarrow_k$ by $(x)\downarrow_0 = 1$ and $(x)\downarrow_k = x(x-1)(x-2)\cdots(x-(k-1))$ for $k \in \mathbb{P}$.

Theorem 3. *(Goldman, Joichi, and White [21])*
Let $B = F(b_1, \ldots, b_n)$ be a Ferrers board. Then

$$\prod_{i=1}^{n}(x + b_i - (i-1)) = \sum_{k=0}^{n} r_{n-k}(B)(x)\downarrow_k . \tag{10}$$

Proof. Since the polynomials are of finite degree, it is enough to prove the (10) holds for all positive integers x. Given a positive integer x, we let B_x be the board which is obtained from $B = F(b_1, \ldots, b_n)$ by attaching x rows of length n below B. For example, if $B = F(1, 2, 4, 4, 6)$, then B_4 is pictured in Figure 3. We shall call the line that separates B from the x rows that we added below B *the bar.* We shall then refer to the cells that lie in B as the cells that lie above the bar and the cells that lie in the x rows that were added below B as the cells below the bar.

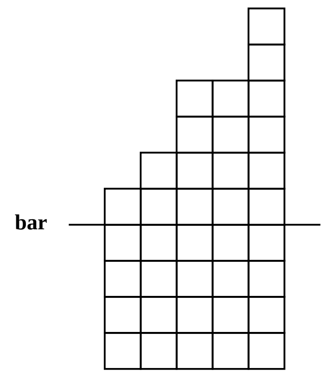

Figure 3: The board B_4 where $B = F(1, 2, 4, 4, 6)$.

Let $\mathcal{N}_n(B_x)$ denote the set of all placements of n nonattacking rooks in B_x. We claim that (10) is the result of two different ways of counting $|\mathcal{N}_n(B_x)|$. That is, if we think of counting the number of ways of placing a rook in each column, reading from left to right, then we clearly have $x+b_1$ ways to place the rook in the first column. Next we have $x+b_2-1$ ways to place the rook in the second column where the -1 comes from the fact that we can not place the rook in the second column in the same row as the rook in the first column. Here we are using the fact that $b_1 \leq b_2$ so that every row in the first column is also a row in the second column. Next we have $x+b_2-2$ ways to place a rook in the third column where the -2 comes from the fact that we can not place a rook in the third column in the rows of the rooks in the first two columns. Continuing on in this way, it is easy to see that

$$|\mathcal{N}_n(B_x)| = \prod_{i=1}^{n}(x + b_i - (i-1)).$$

Next, suppose that we first fix a placement P of $n-k$ non-attacking rooks above the bar in B_x. We claim that there are $(x)\downarrow_k$ ways to extend P to a placement $Q \in \mathcal{N}_n(B_x)$ such that $Q \cap B = P$. That is, we want to count the number of ways to extend P to a placement $Q \in \mathcal{N}_n(B_x)$ by placing an additional k rooks below the bar. If we look at the leftmost available column in which to place a rook below the bar, then there will be x possible cells in which to place it. As we move to the right, the next available column in which to place a rook below the bar will have $x-1$ cells left to place the rook below the bar in that column. Continuing on in this way, we see that number of such Q is $x(x-1)\cdots(x-(k-1)) = (x)\downarrow_k$. Thus we see that

$$\begin{aligned} |\mathcal{N}_n(B_x)| &= \sum_{k=0}^{n} \sum_{P \in \mathcal{N}_{n-k}(B)} (x)\downarrow_k = \sum_{k=0}^{n} (x)\downarrow_k |\mathcal{N}_{n-k}(B)| \\ &= \sum_{k=0}^{n} (x)\downarrow_k r_{n-k}(B) \end{aligned}$$

as desired. □

We have a similar result for file numbers, see Miceli-Remmel [**?**].

Theorem 4. *Let $B = F(b_1, \ldots, b_n)$ be a skyline board. Then*

$$\prod_{i=1}^{n} (x+b_i) = \sum_{k=0}^{n} f_{n-k}(B) x^k. \tag{11}$$

Proof. Since the polynomials are of finite degree, it is enough to prove the (11) holds for all positive integers x.

Let $\mathcal{C}_n(B_x)$ denote the set of all file placements of n rooks in B_x. We claim that (11) is the result of two different ways of counting $|\mathcal{C}_n(B_x)|$. That is, if we think of counting the number of ways of placing a rook in each column, reading from left to right, then we clearly have $x+b_i$ ways to place the rook in the column i. Thus

$$|\mathcal{C}_n(B_x)| = \prod_{i=1}^{n} (x+b_i).$$

Next, suppose that we first fix a file placement P of $n-k$ rooks above the bar in B_x. We claim that there are x^k ways to extend P to a placement $Q \in \mathcal{F}_n(B_x)$ such that $Q \cap B = P$. That is, we want to count the number of ways to extend P to a placement $Q \in \mathcal{N}_n(B_x)$ by placing an additional k rooks below the bar. Clearly there are x ways to place a rook below the bar in any of columns that do not have a rook in P in them. Thus we see that

$$\begin{aligned} |\mathcal{F}_n(B_x)| &= \sum_{k=0}^{n} \sum_{P \in \mathcal{C}_{n-k}(B)} x^k = \sum_{k=0}^{n} x^k |\mathcal{C}_{n-k}(B)| \\ &= \sum_{k=0}^{n} x^k f_{n-k}(B) \end{aligned}$$

□

We end this section with some basic recursions for rook numbers and file numbers for Ferrers boards. First assume that $B = F(b_1, \ldots, b_n)$ is a Ferrers board and that $B^- = F(b_1, \ldots, b_{n-1})$. Then for any $k \geq 1$, we have that

$$r_k(B) = r_k(B^-) + (b_n - k + 1) r_{k-1}(B^-). \tag{12}$$

and

$$f_k(B) = f_k(B^-) + b_n f_{k-1}(B^-). \tag{13}$$

That is, it is easy to see that (12) follows by classifying the elements of $\mathcal{N}_k(B)$ according to whether there is a rook in the last column. Clearly if there is no rook in the last column, then all k rooks must be placed in B^- so that there are $r_k(B^-)$ such rook placements. However, if there is a rook in the last column, then we must place $k-1$ rooks in B^- which we can do in $r_{k-1}(B^-)$ ways. For each placement of $Q \in \mathcal{N}_{k-1}(B^-)$, we can extend Q to a rook placement of k non-attacking rooks in B in $b_n - (k-1)$ ways since we can not place the extra rook in the last column in any of the rows which contain rooks in Q. Thus it follows that there $(b_n - k+1)r_{k-1}(B^-)$ elements of $\mathcal{N}_k(B)$ which contain a rook in the last column. The recursion (13) is proved in the same manner.

1.3 Some special cases of rook and file numbers

In this section, we shall look at some special cases of rook and file numbers. For example, consider the staircase board $St_n = (0, 1, \ldots, n-1)$. Note that in this case, (10) and becomes

$$x^n = \sum_{k=1}^{n} r_{n-k}(St_n)(x)\downarrow_k . \tag{14}$$

Now the Stirling numbers of the second kind $S_{n,k}$ are defined by

$$x^n = \sum_{k=1}^{n} S_{n,k}(x)\downarrow_k \tag{15}$$

from which it immediately follows that

$$S_{n,k} = r_{n-k}(St_n). \tag{16}$$

Now the usual combinatorial interpretation of $S_{n,k}$ is that $S_{n,k}$ is the number of set partitions of $\{1, \ldots, n\}$ into k parts. Thus it follows that the number of placements of $n-k$ non-attacking rooks in St_n is equal to the number of set partitions of $\{1, \ldots, n\}$ into k parts. There is a simple bijection Θ_n which takes the set $\mathcal{SP}_{n,k}$ of set partitions of $\{1, \ldots, n\}$ into k parts onto $\mathcal{N}_{n-k}(St_n)$. That is, suppose that we are given a set partition $P = (C_1, \ldots, C_k) \in \mathcal{SP}_{n,k}$ where $min(C_1) < min(C_2) < \cdots < min(C_k)$. Then for each i, we add no rooks to $\Theta(P)$ if $|C_i| = 1$ and we add rooks in positions $(c_1^i, c_2^i), (c_2^i, c_3^i), \ldots, (c_{s_i-1}^i, c_{s_i}^i)$ if $|C_i| = s_i$ where $s_i \geq 2$ and $C_i = \{c_1^i < \cdots < c_{s_i}^i\}$. For example, if $P = (\{1,3,8\}, \{2\}, \{4,7,8\}, \{5,9\}$, then the rook placement $\Theta_9(P)$ has rooks in positions $(1,3)$, $(3,8)$, $(4,7)$, $(7,8)$, and $(5,9)$. We picture $\Theta_9(P)$ in Figure 4. The inverse map in this case is also easy to describe. That is, given a rook placement $Q \in \mathcal{N}_{n-k}(St_n)$, label the cells which are just to left of St_n in row i with i as pictured in Figure 4. Then $\Theta_n^{-1}(Q)$ is the set partition determined by placing i and j in the same part if there is a rook in cell (i,j) which lies at the intersection of the row labeled i and the column labeled j in Q.

Note that in this case (12) becomes

$$\begin{aligned} S_{n+1,k} &= r_{n+1-k}(St_{n+1}) = r_{n+1-k}(St_n) + (n-(n-k))r_{n-k}(St_n) \\ &= S_{n,k-1} + kS_{n,k} \end{aligned} \tag{17}$$

which is the usual recursion for the Stirling numbers of the second kind.

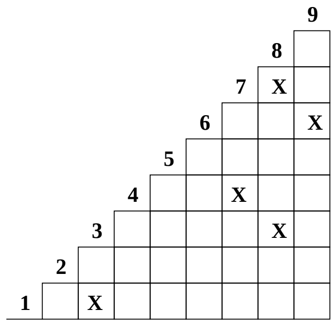

Figure 4: $\Theta_9(P)$.

For $B = St_n$, (11) becomes

$$x(x+1)\cdots(x+n-1) = \sum_{k=1}^{n} f_{n-k}(St_n)x^k. \tag{18}$$

Then replacing x by $-x$ and multiplying both sides by $(-1)^n$, we obtain that

$$(x)\downarrow_n = \sum_{k=1}^{n} (-1)^{n-k} f_{n-k}(St_n)x^k. \tag{19}$$

Now the Stirling numbers of the first kind $s_{n,k}$ are defined by

$$(x)\downarrow_n = \sum_{k=1}^{n} s_{n,k}x^k. \tag{20}$$

from which it immediately follows that

$$s_{n,k} = (-1)^{n-k} f_{n-k}(St_n). \tag{21}$$

Now $c_{n,k} = (-1)^{n-k}s_{n,k}$ is called a signless Stirling number of first kind. This implies that $f_{n-k}(St_n) = c_{n,k}$.

The usual combinatorial interpretation of $c_{n,k}$ is that $c_{n,k}$ is the number permutations of S_n with k-cycles. Thus it follows that the number of file placements of $n-k$ rooks in St_n is equal to the number of permutations of n into k cycles. There is a simple bijection Δ_n which takes the set $\mathrm{PERM}_{n,k}$ of permutations of S_n into k cycles onto $\mathcal{C}_{n-k}(St_n)$. That is, suppose that we are given a permutation $\sigma = (C_1, \ldots, C_k) \in \mathrm{PERM}_{n,k}$. We assume that the cycles of σ which are arranged so that the minimal elements of each cycle C_i is at the left and the cycles are arranged by increasing minimal elements. Then for $i = 1, \ldots, n$ we let $\sigma^{(i)}$ be the permutation of S_i that results by erasing all the elements $i+1, \ldots, n$ in the cycle structure of σ. For example, if $\sigma = (1,7,3)\ (2,4)\ (5,9,8)\ (6)$, then

$$\begin{aligned}
\sigma^{(1)} &= (1), \\
\sigma^{(2)} &= (1)\ (2), \\
\sigma^{(3)} &= (1,3)\ (2), \\
\sigma^{(4)} &= (1,3)\ (2,4), \\
\sigma^{(5)} &= (1,3)\ (2,4)\ (5), \\
\sigma^{(6)} &= (1,3)\ (2,4)\ (5)\ (6), \\
\sigma^{(7)} &= (1,7,3)\ (2,4)\ (5)\ (6), \\
\sigma^{(8)} &= (1,7,3)\ (2,4)\ (5,8)\ (6), \text{ and} \\
\sigma^{(9)} &= (1,7,3)\ (2,4)\ (5,9,8)\ (6)
\end{aligned}$$

This given, we define $\Delta_n(\sigma)$ to be the file placement $f \in \mathcal{C}_{n-k}(St_n)$ such that

1. There is no rook in column i if i is in a 1-cycle in $\sigma^{(i)}$ and
2. there is a rook in row j of column i if i immediately follows j in the cycle structure of $\sigma^{(i)}$.

For example, we picture $\Delta_9(\sigma)$ in Figure 5

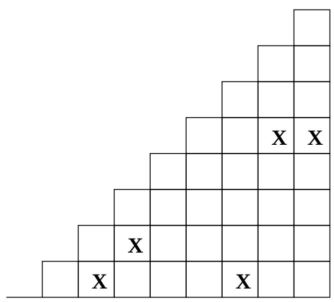

Figure 5: $\Delta_9(\sigma)$ where $\sigma = (1,7,3)\ (2,4)\ (5,9,8)\ (6)$.

Note that in this case (13) becomes

$$\begin{aligned}
c_{n+1,k} &= f_{n+1-k}(St_{n+1}) = f_{n+1-k}(St_n) + n f_{n-k}(St_n) \\
&= c_{n,k-1} + n c_{n,k}
\end{aligned} \tag{22}$$

which is the usual recursion for the $c_{n,k}$'s.

Because the Stirling numbers of the first and second kind are the connections coefficients between the basis $\{x^n : n \geq 0\}$ and $\{x \downarrow_n : n \geq 0\}$, it immediately follows that the infinite matrices $||S_{n,k}||_{n,k\geq 0}$ and $||s_{n,k}||_{n,k\geq 0}$ are inverses of each other.

In fact, we can give a direct combinatorial proof the matrices $||s_{n,k}||$ and $||S_{n,k}||$ are inverses of each other. Our argument is a simplified version of a more general argument due to Remmel and Wachs [47].

That is, if we start with our combinatorial interpretations of $s_{n,k}$ and $S_{n,k}$ in terms of rook placements, then we can give a combinatorial proof of the following for all $0 \leq r \leq n$.

$$\sum_{k=r}^{n} S_{n,k} s_{k,r} = \chi(r = n). \tag{23}$$

Note that if $r = n$, then (23) reduces down to

$$1 = S_{n,n}s_{n,n}. \tag{24}$$

But (24) holds since both $\mathcal{N}_{n-n}(St_n)$ and $\mathcal{C}_{n-n}(St_n)$ consist solely of the empty placement $\mathcal{E}$ so that automatically (24) holds.

Now suppose that $n > r$. Then

$$\begin{aligned}
&\sum_{k=r}^{n} S_{n,k}s_{k,r} \\
&= \sum_{k=r}^{n} |\mathcal{N}_{n-k}(St_n)|(-1)^{k-r}|\mathcal{C}_{k-r}(St_k)| \\
= &\sum_{(\mathcal{P},\mathcal{Q})\in\bigcup_{k=r}^{n}\mathcal{N}_{n-k}(St_n)\times\mathcal{C}_{k-r}(St_k)} sgn(\mathcal{Q})
\end{aligned}$$

where $sgn(\mathcal{Q}) = (-1)^{\text{no. of rooks in } \mathcal{Q}}$. Then consider the elements

$$(\mathcal{P},\mathcal{Q}) \in \bigcup_{k=r}^{n} \mathcal{N}_{n-k}(St_n) \times \mathcal{C}_{k-r}(St_k).$$

We can partition these elements into three classes.

Class I. There is a rook of $\mathcal{P}$ in the last column of St_n.

Class II. There is no rook of $\mathcal{P}$ in the last column of St_n, but there is a rook of $\mathcal{Q}$ in the last column of St_k.

Class III. There is no rook of $\mathcal{P}$ in the last column of St_n and there is no rook of $\mathcal{Q}$ in the last column of St_k.

Next we define a sign-reversing bijection f from Class I to Class II. Suppose that we are given an element $(\mathcal{P},\mathcal{Q}) \in \mathcal{N}_{n-k}(St_n) \times \mathcal{C}_{k-r}(St_k)$ in Class I. We will say that a rook r in $\mathcal{P}$ attacks all the cells in its row of St_n that lie strictly to the right of r. Thus since there are a total of $n-k-1$ rooks in $\mathcal{P}$ to the left of the last column of St_n, there are exactly $k = (n-1)-(n-k-1)$ cells of the last column of St_n that are not attacked by some rook of $\mathcal{P}$ in its row which lies strictly to the left of the last column of St_n. Then define $f((\mathcal{P},\mathcal{Q})) = (\mathcal{P}',\mathcal{Q}')$ where

(i) $\mathcal{P}'$ is the result of taking the placement $\mathcal{P}$ and removing the rook in the last column of St_n and

(ii) $\mathcal{Q}'$ is the result of adding an extra column of height k to the right of the placement $\mathcal{Q}$ and placing a rook f_k in that column which is in row t if the rook r_n in $\mathcal{P}$ in the last column of St_n was in the t-th cell, reading from bottom to top, which was not attacked by a rook in $\mathcal{P}$ to the left of r_n.

See Figure 6 for an example of this map when $n = 6$, $k = 3$ and $r = 2$. Clearly $sgn(\mathcal{Q}) = (-1)^{k-r} = -sgn(\mathcal{Q}') = (-1)^{k-r+1}$ so that f is a sign preserving bijection which for each $r \leq k \leq n$ maps the elements of $\mathcal{N}_{n-k}(St_n)\times\mathcal{C}_{k-r}(St_k)$ in Class I to the elements $\mathcal{N}_{n-k-1}(St_n)\times\mathcal{C}_{k+1-r}(St_{k+1})$ in Class II. Moreover f^{-1} is easily defined. That is, if $(\mathcal{P}',\mathcal{Q}') \in \mathcal{N}_{n-k-1}(St_n)\times\mathcal{C}_{k+1-r}(St_{k+1})$ is in Class II and the rook in the last column of $\mathcal{Q}'$ is in row t, then $f^{-1}(\mathcal{P}',\mathcal{Q}')) = (\mathcal{P},\mathcal{Q})$ where $\mathcal{Q}$ is results from $\mathcal{Q}'$ by removing the last column of St_{k+1} and $\mathcal{P}$ results from $\mathcal{P}'$ by adding a rook in the last column of St_n in the t-th cell from the bottom which is not attacked by any rook in $\mathcal{P}'$. Thus f shows that

$$\sum_{k=r}^{n} \sum_{(\mathcal{P},\mathcal{Q})\in\mathcal{N}_{n-k}(St_n)\times\mathcal{C}_{k-r}(St_k)} sgn(Q) = \sum_{k=r}^{n} \sum_{(\mathcal{P},\mathcal{Q})\in Class\ III} sgn(\mathcal{Q}).$$

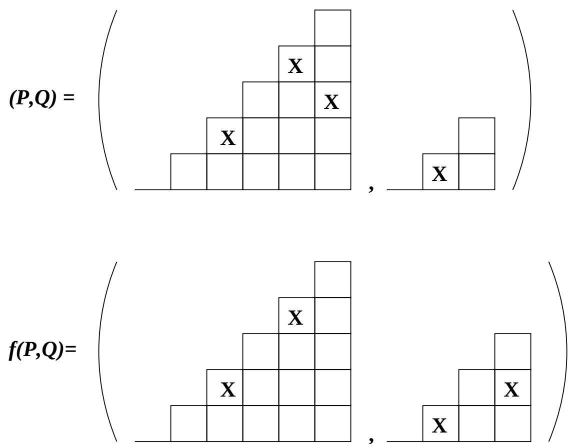

Figure 6: An example of the map f from Class I to Class II.

Note if $r = 0$, then there are no elements in Class III since every element of $(\mathcal{P},\mathcal{Q}) \in \mathcal{N}_{n-k}(St_n)\times\mathcal{C}_{k-0}(St_k)$ has a rook of $\mathcal{Q}$ in the last column of St_k. Thus if $r = 0$, then f shows that $\sum_{k=0}^{n} S_{n,k}s_{k,0} = 0$. Finally if $r \geq 1$, then there is a bijection g which maps Class III onto $\bigcup_{k=r-1}^{n-1}\mathcal{N}_{n-1-k}(St_{n-1})\times\mathcal{C}_{k-(r-1)}(St_k)$. That is, if $(\mathcal{P},\mathcal{Q})$ is in Class III, then $g((\mathcal{P},\mathcal{Q})) = (\mathcal{P}'',\mathcal{Q}'')$ where $\mathcal{P}''$ is obtained from $\mathcal{P}$ by removing its last column and $\mathcal{Q}''$ is obtained from $\mathcal{Q}$ by removing its last column. See Figure 7 for an example.

Thus if $r \geq 1$, then our bijections f and g show that

$$\sum_{k=r}^{n} S_{n,k}s_{k,r} = \sum_{k=r-1}^{n-1} S_{n-1,k}s_{k,r-1} = \chi(r-1 = n-1) \tag{25}$$

where the last equality follows by induction. Thus we have proved that

$$\sum_{k=r}^{n} S_{n,k}s_{k,r} = \chi(r = n)$$

as desired.

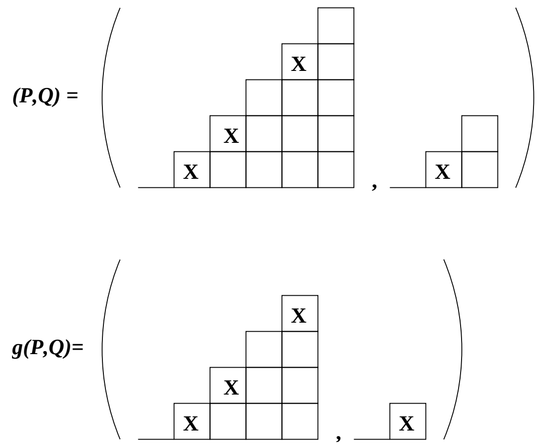

Figure 7: An example of the map g.

The fact that the infinite matrices $||S_{n,k}||_{n,k\geq 0}$ and $||s_{n,k}||_{n,k\geq 0}$ are inverses of each other is really just a consequence of the form of the recursions for the $S_{n,k}$'s and $s_{n,k}$'s. Since we will see similar recursions of this form latter on when we discuss q-analogues and p,q-analogues of our results, we will pause briefly to show that this fact is direct consequence of a more general theorem. The following theorem is simplified version of a result of Milne [46].

Theorem 5. *(Milne [46])*
Let R be a ring and $x_0, x_1, \ldots$ be any sequence of elements from R. Define an infinite lower triangular matrix $||A_{n,k}||_{n,k\geq 0}$ where $A_{n,k}$ are defined by the following recursions:

$$\begin{aligned} A_{0,0} &= 1, \quad A_{n,k} = 0 \text{ if } n < k \text{ or } k < 0 \\ A_{n+1,k} &= A_{n,k-1} + x_k A_{n,k} \text{ for } 0 \leq k \leq n+1. \end{aligned}$$

Then the entries of $A^{-1} = B = ||B_{n,k}||_{n,k\geq 0}$ are defined by the following recursions:

$$\begin{aligned} B_{0,0} &= 1, \quad B_{n,k} = 0 \text{ if } n < k \text{ or } k < 0 \\ B_{n+1,k} &= B_{n,k-1} - x_n B_{n,k} \text{ for } 0 \leq k \leq n+1. \end{aligned}$$

Proof. We must show that for all $n, k \geq 0$,

$$\sum_{\ell} B_{n,\ell} A_{\ell,k} = \chi(n = \ell). \tag{26}$$

To this end, we shall consider two generating functions.

$$\begin{aligned} A_k(t) &= \sum_{n\geq k} A_{n,k} t^n \text{ and} \\ B_n(t) &= \sum_{k=0}^{n} B_{n,k} t^k. \end{aligned}$$

Note that by definition $B_0(x) = 1$. Now it is easy to prove by induction that $A_{n,0} = x_0^n$. That is, $A_{0,0} = 1 = x_0^0$ and if $A_{n,0} = x_0^n$, then

$$A_{n+1,0} = A_{n,-1} + x_0 A_{n,0} = 0 + x_0 x_0^n = x_0^{n+1}.$$

Thus

$$A_0(t) = \sum_{n \geq 0} x_0^n t^n = \frac{1}{1 - x_0 t}. \tag{27}$$

Hence, for $k \geq 1$,

$$\begin{aligned}
A_k(t) &= \sum_{n \geq k} A_{n,k} t^n \\
&= \sum_{n \geq k} (A_{n-1,k-1} + x_k A_{n-1,k}) t^n \\
&= t \sum_{n \geq k} A_{n-1,k-1} t^{n-1} + x_k t \sum_{n \geq k} A_{n-1,k} t^{n-1} \\
&= t A_{k-1}(t) + x_k t A_k(t).
\end{aligned}$$

But then we have that for all $k \geq 1$.

$$A_k(t) = \frac{t}{(1 - x_k t)} A_{k-1}(t). \tag{28}$$

Iterating (28) and using (27), we see that

$$A_k(t) = \frac{t^k}{(1 - x_0 t)(1 - x_1 t) \cdots (1 - x_k t)}. \tag{29}$$

Similarly for $n \geq 1$,

$$\begin{aligned}
B_n(t) &= \sum_{k=0}^{n} B_{n,k} t^k \\
&= \sum_{k=0}^{n} (B_{n-1,k-1} - x_{n-1} B_{n-1,k}) t^k \\
&= t \sum_{k=0}^{n} B_{n-1,k-1} t^{k-1} - x_{n-1} \sum_{k=0}^{n} B_{n-1,k} t^k \\
&= t B_{n-1}(t) - x_{n-1} B_{n-1}(t) = (t - x_{n-1}) B_{n-1}(t).
\end{aligned}$$

Hence, for all $n \geq 1$,

$$B_n(t) = (t - x_{n-1}) B_{n-1}(t). \tag{30}$$

Iterating (30) and using the fact that $B_0(t) = 1$, we see that for all $n \geq 1$,

$$B_n(t) = (t - x_0)(t - x_1) \cdots (t - x_{n-1}). \tag{31}$$

But then for $n \geq 1$,

$$\begin{aligned}
B_{n,k} &= (t - x_0)(t - x_1) \cdots (t - x_{n-1})|_{t^k} \\
&= (\frac{1}{t} - x_0)(\frac{1}{t} - x_1) \cdots (\frac{1}{t} - x_{n-1})|_{t^{-k}} \\
&= \frac{1}{t^n}((1 - x_0 t)(1 - x_1 t) \cdots (1 - x_{n-1} t))|_{t^{-k}} \\
&= (1 - x_0 t)(1 - x_1 t) \cdots (1 - x_{n-1} t))|_{t^{n-k}}.
\end{aligned}$$

We are now in a position to prove (26). That is, when $n = 0$,

$$\sum_\ell B_{0,\ell}A_{\ell,k} = B_{0,0}A_{0,k} = \chi(k = 0)$$

by our initial conditions in the definitions of $B_{n,k}$ and $A_{n,k}$. For $n \geq 1$,

$$\begin{aligned}
&\sum_\ell B_{n,\ell}A_{\ell,k} = \\
&\sum_\ell ((1-x_0t)\cdots(1-x_{n-1}t))\,|_{t^{n-\ell}} \left(\frac{t^k}{(1-x_0t)\cdots(1-x_kt)}\right)_{t^\ell} = \\
&((1-x_0t)\cdots(1-x_{n-1}t)) \left(\frac{t^k}{(1-x_0t)\cdots(1-x_kt)}\right) |_{t^n}. \qquad (32)
\end{aligned}$$

But it is easy to see that (32) is equal to

$$\begin{aligned}
&(1-tx_{k+1})\cdots(1-tx_{n-1})t^k|_{t^n} = 0 \text{ if } n > k, \\
&\frac{t^k}{1-tx_k}|_{t^k} = 1 \text{ if } n = k, \text{ and} \\
&\frac{t^k}{(1-tx_n)\cdots(1-tx_k)}|_{t^n} = 0 \text{ if } n < k.
\end{aligned}$$

Thus (32) is equal to $\chi(n = k)$ as desired. □

Note that since choosing $x_k = k$ in Theorem 5 gives that $A_{n,k} = S_{n,k}$, we have the following corollary as a consequence of the (29).

Corollary 1. *For all $k \geq 0$,*

$$\sum_{n\geq k} S_{n,k}t^n = \frac{t^k}{(1-x)(1-2x)\cdots(1-kx)} \qquad (33)$$

Exercises

(1) Let $F_k(x) = \sum_{n\geq k} S_{n,k}\frac{x^n}{n!}$. Use the recursion satisfied by the $S_{n,k}$'s to show

$$\frac{d}{dx}(F_k(x)) = F_{k-1}(x) + kF_k(x). \qquad (34)$$

show that (34) plus the initial condition that $S_{k,k} = 1$ completely determines $F_k(x)$ if we know $F_{k-1}(x)$. Then use this fact to prove by induction that

$$F_k(x) = \frac{1}{k!}(e^x - 1)^k. \qquad (35)$$

(2) Let $G_k(x) = \sum_{n\geq k} c_{n,k}\frac{x^n}{n!}$. Use the recursion satisfied by the $c_{n,k}$'s to show

$$\frac{d}{dx}(G_k(x)) = G_{k-1}(x) + xG_k(x). \qquad (36)$$

show that (36) plus the initial condition that $c_{k,k} = 1$ completely determines $G_k(x)$ if we know $G_{k-1}(x)$. Then use this fact to prove by induction that

$$G_k(x) = \frac{1}{k!}\left(ln\left(\frac{1}{1-x}\right)\right)^k. \tag{37}$$

(3) Let $B_n = \sum_{k=1}^{n} S_{n,k}$ denote the number of set partitions of $\{1,\ldots,n\}$. The B_n are called the Bell numbers.

(a) Show that the Bell numbers satisfy the recursion

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k. \tag{38}$$

(b) Show that $\sum_{n\geq 0} B_n \frac{x^n}{n!} = e^{e^x - 1}$.

Another interesting board is the board $\mathcal{L}_n$ which is the Ferrers board which consists of n columns of height $n-1$. We shall call $\mathcal{L}_n$ the Laguerre board and let $L_{n,k} = r_{n-k}(\mathcal{L}_n)$. In that case, (10) becomes

$$(x)\uparrow_n = \sum_{k=1}^{n} L_{n,k}(x)\downarrow_k \tag{39}$$

where we let $(x)\uparrow_0 = 1$ and $(x)\uparrow_n = x(x+1)\cdots(x+n-1)$ for $n \geq 1$. Thus the $L_{n,k}$'s are the connection coefficients between rising factorial basis the falling factorial basis of the polynomial ring $Q[x]$. The $L_{n,k}$ are called the *Lah numbers* after Ivo Lah [**?**]. It is easy to count the number of placements $\mathcal{P}$ of $n-k$ non-attacking rooks in $\mathcal{L}_n$. That is, we can pick the $k-1$ rows that do not contain a rook in $\mathcal{P}$ in $\binom{n-1}{k-1}$ ways. Then let $1 < R_1 < \ldots R_{n-k} \leq n-1$ be the rows which will contain rooks in $\mathcal{P}$. It is clear that we have n ways to pick where a rook goes in R_1, $n-1$ ways to pick where the rook in goes in R_2, etc.. Thus

$$L_{n,k} = (n)\downarrow_{n-k} \binom{n-1}{k-1} = \frac{n!}{k!}\binom{n-1}{k-1}. \tag{40}$$

Exercises

(4) Show that the Lah numbers $L_{n,k}$ satisfy the following recursions:

$$L_{n+1,k} = L_{n,k-1} + (k+n)L_{n,k} \tag{41}$$

The standard combinatorial interpretation of the Lah numbers is that $L_{n,k}$ is the number of ways to place n labeled balls in k unlabeled tubes. Let $\mathcal{T}_{n,k}$ be the set of placements of n balls labeled $1,\ldots,n$ in k unlabeled tubes. It is easy to describe a bijection $F : \mathcal{T}_{n,k} \to \mathcal{N}_{n-k}(\mathcal{L}_n)$. That is, suppose that we start with an element $Q \in \mathcal{T}_{n,k}$. We order the tubes by increasing

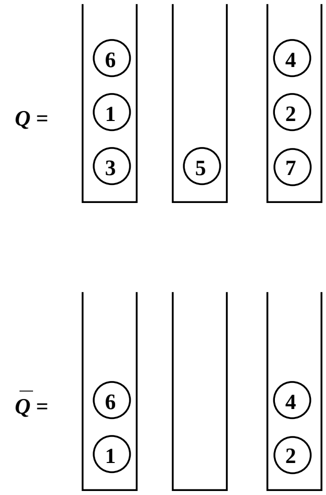

Figure 8: A placement of 7 labeled balls in 3 unlabeled tubes.

bottom elements and let $\bar{Q}$ be the configuration that results by removing the bottom balls in each tube as indicated in Figure 8.

Then let $s_1, s_2, \ldots, s_k$ be the number of balls in the tubes of $\bar{Q}$ reading from left to right. If $s_1 = 0$ so that the first tube of $\bar{Q}$ is empty, then bottom row of $F(Q)$ will be empty. Otherwise if $i_1^1, \ldots, i_{s_1}^1$ are the elements in the first tube of $\bar{Q}$, reading from bottom to top, then we place a rook in row t in column i_t^1 for $t = 1, \ldots, s_1$ and then we ensure that row $s_1 + 1$ is empty. Thus if there are s_1 balls in the first tube of $\bar{Q}$, then we will have determined the placement or non-placement of rooks in the first $s_1 + 1$ rows of $F(Q)$. In general, the first j tubes of $\bar{Q}$ will determine the placement or non-placement of rooks in rows $1, \ldots, s_1 + \cdots + s_j + j$. Then if $s_{j+1} = 0$, we ensure that row $1 + s_1 + \cdots + s_j + j$ is empty in $F(Q)$. Otherwise, if $i_1^{j+1}, \ldots, i_{s_{j+1}}^{j+1}$ are the elements in the $j+1$-st tube of $\bar{Q}$, reading from bottom to top, then we place a rook in row $t + s_1 + \cdots + s_j + j$ in column i_t^{j+1} for $t = 1, \ldots, s_{j+1}$. If $j + 1 < k$ we ensure that row $s_{j+1} + 1 + s_1 + \cdots + s_j + j$ is empty and go on to tube $j + 2$. However if $j + 1 = k$, then we simply stop. For example,the image under F of the Q pictured in Figure 8 is given in Figure 9.

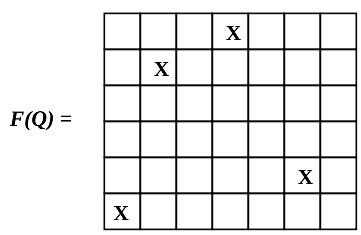

Figure 9: $F(Q)$.

1.4 Rook equivalence for Ferrers boards

Given two rook boards B_1 and B_2, we say that B_1 and B_2 are *rook equivalent* if $r_k(B_1) = r_k(B_2)$ for all $k \geq 1$. In this section, we shall prove a number of results about the rook equivalence for Ferrers boards.

If $F(b_1, \ldots, b_n)$ is a Ferrers board, we let $\mathcal{S}(b_1, \ldots, b_n)$ be the multiset $\{b_1, b_2 - 1, \ldots, b_n - (n-1)\}$. For example, if $\mathcal{S}(0,0,1,1,2,2) = \{0,-1,-1,-2,-2,-3\}$. Then we have the following characterization of rook equivalence for Ferrers boards due to Foata and Schützenberger [14].

Theorem 6. *Let $B_1 = F(a_1, \ldots, a_n)$ and $B_2 = F(b_1, \ldots, b_n)$ be two Ferrers boards contained in $[n] \times [n]$. Then B_1 and B_2 are rook equivalent if and only if $\mathcal{S}(a_1, \ldots, a_n)$ and $\mathcal{S}(b_1, \ldots, b_n)$ are equal as multisets.*

Proof. By Theorem 3,

$$\begin{aligned}\prod_{i=1}^{n}(x + a_i - (i-1)) &= \sum_{k=0}^{n} r_{n-k}(B_1)(x)\downarrow_k \text{ and} \\ \prod_{i=1}^{n}(x + b_i - (i-1)) &= \sum_{k=0}^{n} r_{n-k}(B_2)(x)\downarrow_k .\end{aligned}$$

Thus if $\mathcal{S}(a_1, \ldots, a_n) = \mathcal{S}(b_1, \ldots, b_n)$, then we have

$$\sum_{k=0}^{n} r_{n-k}(B_1)(x)\downarrow_k = \sum_{k=0}^{n} r_{n-k}(B_2)(x)\downarrow_k$$

which clearly implies that $r_{n-k}(B_1) = r_{n-k}(B_2)$ for all $k = 0, \ldots, n$ since $\{(x)\downarrow_k : k \geq 0\}$ is a basis for $\mathbb{Q}[x]$. Similarly, if the B_1 and and B_2 are rook equivalent, then

$$\prod_{i=1}^{n}(x + a_i - (i-1)) = \prod_{i=1}^{n}(x + b_i - (i-1))$$

which implies $\mathcal{S}(a_1, \ldots, a_n)$ and $\mathcal{S}(b_1, \ldots, b_n)$ are equal as multisets. □

We say that $F(b_1, \ldots, b_n)$ is a *strictly increasing* Ferrers board if the non-zero columns of B are strictly increasing from left to right, i.e. if there is a $k \geq 0$ such that $b_i = 0$ for $i < k$ and $0 < b_k < b_{k+1} < \cdots < b_n$. Then Foata and Schützenberger [] proved the following.

Theorem 7. *(Foata-Schützenberger [])*
For any Ferrers board B_1, there is a unique increasing Ferrers board B_2 which is rook equivalent to B_1.

Proof. For any Ferrer board B of size n, add enough 0 columns to the left of B so that $B = F(b_1, \ldots, b_{n+1})$. For example, if $B = F(2,2,3)$, then $|B| = 7$ and we let $B = F(0,0,0,0,0,2,2,3)$. In such a situation, we claim that $b_i - (i-1) \leq 0$ for all $i = 1, \ldots, n+1$. That is, $b_1 = 0$ since B has at most n non-zero columns. Note that for any pair of positive integers x and y, $xy \geq x + y - 1$. Thus if $b_i \geq i$, then B would have at least $i(n+1-(i-1)) = i(n+2-i) > n+1$ squares. Thus $b_i \leq i-1$ for all $i = 2, \ldots, n+1$.

Now suppose that $B_1 = F(a_1, \ldots, a_{n+1})$ where $|B| = n$. Then by our remarks above $\mathcal{S}(a_1, \ldots, a_{n+1})$ consists only of numbers which are less that 0. Moreover since B is a Ferrers board, then $b_{i+1} - i \geq b_i - (i-1) - 1$ so that we can only decrease by a maximum of 1 in going form $b_i - (i-1)$ to $b_{i+1} - i$. Now suppose that $-k$ is the minimum element in $\mathcal{S}(a_1, \ldots, a_{n+1})$. Then it must be the case that $0, -1, -2, \ldots, -k$ are also in $\mathcal{S}(a_1, \ldots, a_{n+1})$. Thus we can rearrange elements of $\mathcal{S}(a_1, \ldots, a_{n+1})$ to obtain a sequence

$$(s_1, \ldots, s_{n+1}) = (0, -1, -2, \ldots, -k \leq s_{k+2} \leq \cdots \leq s_{n+1}.$$

But then if $b_i = s_i + (i-1)$ it is easy to see that $b_i = 0$ for $i \leq k+1$ and $b_{k+2} < \cdots < b_{n+1}$ so that $F(b_1, \ldots, b_{n+1})$ is an increasing Ferrers board with $\mathcal{S}(b_1, \ldots, b_{n+1}) = \mathcal{S}(a_1, \ldots, a_{n+1})$, Hence, B_1 is rook equivalent to B_2 by Theorem 6. For example, if $B = F(0,0,0,0,0,2,2,3)$,

$$\mathcal{S}(0,0,0,0,0,2,2,3) = \{0,-1,-2,-3,-4,-3,-4,-4\}$$

so we would consider the rearrangement $(s_1, \ldots, s_8) = (0,-1,-2,-3,-4,-4,-4,-3)$. Then $(b_1, \ldots, b_8) = (0,0,0,0,0,1,2,4)$.

For uniqueness, suppose that $B_2 = F(b_1, \ldots, b_{n+1})$ is increasing board with $|B_2| = n$ and $k \geq 0$ is such that $b_i = 0$ for $i \leq k+1$ and $0 < b_{k+2} < \cdots < b_{n+1}$. Then it is easy to see that sequence $(b_1, b_2 - 1, \ldots, b_{n+1} - n)$ is of the form $(0 \ -1 \ -2 \ldots -k \leq s_{k+2} \leq \cdots \leq s_{n+1})$ so that there is only one increasing Ferrers board $B_2 = F(b_1, \ldots, b_{n+1})$ with $\mathcal{S}(a_1, \ldots, a_{n+1}) = \mathcal{S}(b_1, \ldots, b_{n+1})$. $\square$

Goldman, Joichi, and White [21] proved the following formula for the number of Ferrers boards which are rook equivalent to a given Ferrers board. As in the proof the Theorem 7, we can write any Ferrers board B of size n in the form $B = F(b_1, \ldots, b_{n+1})$ by adding an appropriate number of 0's to the sequence for B.

Theorem 8. *Suppose that B is a Ferrers board of size n and $B = F(b_1, \ldots, b_{n+1})$ where $|B| = n$. Suppose that $\mathcal{S}(b_1, \ldots, b_{n+1})$ is a rearrangement of $0^{a_0}(-1)^{a_1} \ldots (-k)^{a_k}$. Then the number of Ferrers boards which are rook equivalent to B equals*

$$\binom{a_0 + a_1 - 1}{a_1}\binom{a_1 + a_2 - 1}{a_2} \cdots \binom{a_{k-1} + a_k - 1}{a_k}.$$

Proof. By Theorem 6, if $B^* = (b_1^*, \ldots, b_{n+1}^*)$ is rook equivalent to B, then $\mathcal{S}(b_1^*, \ldots, b_{n+1}^*) = (d_1, \ldots, d_{n+1})$ must be a rearrangement of $0^{a_0}(-1)^{a_1} \ldots (-k)^{a_k}$ that starts with 0 and satisfies $d_i - d_{i+1} \geq -1$. Thus we need only count the number of such sequences $(d_1, \ldots, d_{n+1})$. Now if we consider the 0's and the -1's in such a sequence it is easy to see that the -1's must be placed in spaces following the 0's. Since there are $a_0 - 1$ such spaces, there are $\binom{a_0+a_1-1}{a_1}$ ways to place the -1's in such spaces. That is any rearrangement of $0^{a_0-1}(-1)^{a_1}$ determines such an arrangement. Similarly, having placed the 0's and -1's, then the -2's must be in the spaces following the -1's. Since there are $a_1 - 1$ such spaces, there are $\binom{a_1+a_2-1}{a_2}$ ways to place the -2's in such spaces. Continuing on in this way yield the desired formula. $\square$

1.5 Applications to Permutation Enumeration

In this section, we shall give several applications of the simple rook theory theorems proved in the previous section to permutation enumeration. The basic idea is that we think of a board $B \subseteq [n] \times [n]$ as a set of "forbidden positions" for permutations. Thus $H_0(B)$ is the number of permutations which avoid these forbidden positions.

Example 1: Derangements

Given a permutation $\sigma = \sigma_1 \ldots \sigma_n \in S_n$, we say that i is fixed point of σ if $\sigma_i = i$. We let $\mathbb{D}_n$ denote the set of permutations with no fixed points. A permutation $\sigma \in \mathbb{D}_n$ is called a *derangement* and we let $D_n = |\mathbb{D}_n|$ denote the number of derangements in S_n.

Now consider the diagonal board $Diag_n = \{(i,i) : i = 1, \ldots, n\} \subseteq [n] \times [n]$. Then clearly, $H_{0,n}(Diag_n) = D_n$ and, more generally, $H_{k,n}(Diag_n)$ is the number of permutations with exactly k fixed points. It is easy to see that $r_k(Dg_n) = \binom{n}{k}$ so that by Riordan and Kaplansky's theorem, Theorem 1, we have that

$$\sum_{k=0}^{n} H_{k,n}(Dg_n)x^k = \sum_{k=0}^{n} \binom{n}{k}(n-k)!(x-1)^k. \tag{42}$$

Taking the coefficients of x^k on both sides of (42), we see that

$$H_{k,n}(Dg_n) = \sum_{s=k}^{n}(-1)^{s-k}\binom{s}{k}\binom{n}{s}(s-k)! \tag{43}$$

Thus, in particular,

$$D_n = \sum_{s=0}^{n}(-1)^{s}\binom{n}{s}(s-k)!$$

or, equivalently,

$$\frac{D_n}{n!} = \sum_{s=0}^{n}\frac{(-1)^s}{s!} \tag{44}$$

Multiplying both sides of (44) by t^n and summing, we find that the exponential generating function of derangements is given by

$$\begin{aligned}
\sum_{n\geq 0} D_n \frac{t^n}{n!} &= \sum_{n\geq 0} t^n \sum_{s=0}^{n}\frac{(-1)^s}{s!} \\
&= (\sum_{s\geq 0}\frac{(-1)^s}{s!}t^s)(\sum_{n\geq 0} t^n) \\
&= \frac{e^{-t}}{1-t}
\end{aligned}$$

Equation (44) also immediately implies a simple recursion for D_n. That is, for $n \geq 1$,

$$\begin{aligned}
\frac{D_n}{n!} &= \sum_{s=0}^{n}\frac{(-1)^s}{s!} \\
&= \sum_{s=0}^{n-1}\frac{(-1)^s}{s!} + \frac{(-1)^n}{n!} \\
&= \frac{D_{n-1}}{(n-1)!} + \frac{(-1)^n}{n!}.
\end{aligned}$$

Multiplying both sides of the above equation by $n!$ yields the following recursion.

$$D_n = nD_{n-1} + (-1)^n \text{ for all } n \geq 1. \tag{45}$$

Applying (45) twice we get the following recursion

$$\begin{aligned} D_n &= nD_{n-1} + -1^n \\ &= (n-1)D_{n-1} + D_{n-1} + -1^n \\ &= (n-1)D_{n-1} + (n-1)D_{n-2} + (-1)^{n-1} + -1^n \\ &= (n-1)D_{n-1} + (n-1)D_{n-2} \text{ for all } n \geq 2. \end{aligned} \tag{46}$$

We note (45) has a simple combinatorial proof. That is, we simple simply partition the set of derangements $\sigma \in \mathbb{D}_n$ into two classes depending on whether n lies in a 2-cycle in σ or n lies in a k-cycle in σ where $k \geq 3$. If n is in a two cycle (i,n) in σ, then we can remove this two cycle and the remaining cycle structured can be renumbered to give the cycle structure of a derangment in $\mathbb{D}_{n-2}$. Since we have $n-1$ choices for i, it is easy to see that there are $(n-1)D_{n-2}$ $\sigma \in \mathbb{D}_n$ where n lies in a 2-cycle in σ. Next suppose that n lies in a k-cycle in $\sigma \in \mathbb{D}$ where $k \geq 3$. Write the cycles of σ so that the smallest element in each cycle in on the left and we order the cycles by increasing smallest elements. For example, suppose that $\sigma = (1,4,6,5)(2,11,8,3)(6,9,10)$. We can remove n from σ to obtain the cycle structure of a permutation $\tau \in \mathbb{D}_{n-1}$. In our example, $\tau = (1,4,6,5)(2,8,3)(6,9,10)$. It is then easy to see that if we insert n after element in the cycle structure of τ, we will obtain $n-1$ permutations $\bar{\sigma} \in \mathbb{D}_n$ such that n lies in a k-cycle in $\bar{\sigma} \in \mathbb{D}$ where $k \geq 3$ and where removing n from $\bar{\sigma}$ gives τ. Thus the number of $\sigma \in \mathbb{D}_n$ such that n lies in a k-cycle in σ where $k \geq 3$ is equal to $(n-1)D_{n-1}$.

Example 2. Counting permutations by the number of k-excedances or the number of k-drops.

Given a permutation $\sigma = \sigma_1 \dots \sigma_n \in S_n$ and $k \geq 1$, we let

$$\begin{array}{ll} \mathrm{Des}(\sigma) = \{i : \sigma_i > \sigma_{i+1}\} & \mathrm{des}(\sigma) = |\mathrm{Des}(\sigma)| \\ \mathrm{Des}_k(\sigma) = \{i : \sigma_i - \sigma_{i+1} = k\} & \mathrm{des}_k(\sigma) = |\mathrm{Des}_k(\sigma)| \\ \mathrm{Exc}(\sigma) = \{i : i < \sigma_i\} & \mathrm{exc}(\sigma) = |\mathrm{Exc}(\sigma)| \\ \mathrm{Exc}_k(\sigma) = \{i : \sigma_i - i = k\} & \mathrm{exc}_k(\sigma) = |\mathrm{Exc}_k(\sigma)| \end{array}.$$

Elements of $\mathrm{Des}(\sigma)$ are called the *descents* of σ and pairs (σ_i, σ_{i+1}) with $i \in \mathrm{Des}(\sigma)$ are called *descent pairs* of σ. Similarly, elements of $\mathrm{Exc}(\sigma)$ are called the *excedances* of σ and pairs (i, σ_i) with $i \in \mathrm{Exc}(\sigma)$ are called *excedance pairs* of σ. For $k \geq 1$, we shall call the elements of $\mathrm{Des}_k(\sigma)$ *k-decents* and the pairs (σ_i, σ_{i+1}) with $i \in \mathrm{Des}_k(\sigma)$ *k-descent pairs* of σ. Similarly, elements of $\mathrm{Exc}_k(\sigma)$ are called the *k-excedances* of σ and pairs (i, σ_i) with $i \in \mathrm{Exc}_k(\sigma)$ are called *k-excedance pairs* of σ.

Foata's First Fundamental Transformation

There is a fundamental bijection, called Foata's First Transformation [15], which is a bijection $\Phi : S_n \longrightarrow S_n$ which shows that distributions of descents and excedances in over S_n are equal. For a permutation $\sigma = \sigma_1 \dots \sigma_n$, we say that σ_j is a *left-to-right maximum (minimum)* if $\sigma_j > \sigma_i$ ($\sigma_j < \sigma_i$) for all $i < j$. Foata's transformation can most easily be explained with an example. Let $\omega = 61437258 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 1 & 4 & 3 & 7 & 2 & 5 & 8 \end{pmatrix}$. This permutation has three excedances pairs $(1,6)$, $(3,4)$, and $(5,7)$. The first step in Foata's transformation is to write ω in cycle form: $(162)(34)(57)(8)$. Next, write each cycle with largest element last, and order the cycles by increasing largest element: $(34)(216)(57)(8)$. Finally, to compute $\Phi(\omega)$, reverse each cycle and erase the parentheses: $\Phi(\omega) = 43612758$. In this example the descents of $\Phi(\omega)$ are $43, 61$, and 75. In general, it is not hard to see that (i,j) is a descent pair of $\Phi(\omega)$ if and only if (j,i) is an excedance of ω. To go backwards, given $\sigma = 43612758$, cut before each left-to-right maxima: $43|612|75|8$, then reverse each block to get the cycles of $\Phi^{-1}(\sigma)$: $(34)(216)(57)(8)$. Thus for each $k \geq 1$, (i,j) is a k-descent pair of $\Phi(\omega)$ if and only if (j,i) is a k-excedance pair of ω.

Thus Foata's first fundamental transformation shows for all $k, n \geq 1$,

$$\sum_{\sigma \in S_n} x^{\mathrm{des}_k(\sigma)} = \sum_{\sigma \in S_n} x^{\mathrm{exc}_k(\sigma)}. \tag{47}$$

Note that $\mathrm{Exc}_k(\sigma)$ makes sense when $k = 0$ because in that case, $\mathrm{Exc}_0(\sigma)$ is just the set of fixed points of σ. Thus for any $k \geq 0$, we define

$$P_{n,k}(x) = \sum_{\sigma \in S_n} x^{\mathrm{exc}_k(\sigma)} = \sum_{s=0}^{n-k} P_{n,k,s} x^s. \tag{48}$$

By convention, we also define $P_{0,0,0} = 1$. Note for example, $P_{n,0,0}$ is the number derangements.

Rakotondrajao [49] was the first to study the numbers $P_{n,k,s}$. In fact, Rakotondrajao called elements of $\mathrm{Exc}_k(\sigma)$ k-successions. He proved a number of basic recursions for the numbers $P_{n,k,s}$ as well as gave explicit expressions for some simple generating functions involving the $P_{n,k,s}$. Later Liese and Remmel [41] studied q-analogues of the $P_{n,k,s}$s, gave the alternative interpretation of $P_{n,k,s}$ in terms of k-descents, and showed that the polynomial $P_{n,k}(x)$ is just a special case of a hit polynomial. That is, let $B_{n,k}$ be the board contained in $[n] \times [n]$ which consists of the diagonal connecting $(1, 1+k)$ and $(n-k, n)$. For example, the board $B_{7,2}$ is pictured in figure 10.

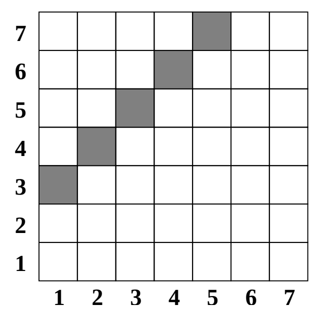

Figure 10: The board $B_{7,2}$.

It is then easy to see that the number of $\sigma \in S_n$ with s k-excedances is the s-th hit number of $B_{n,k}$, i.e. $P_{n,k,s} = H_s(B_{n,k})$. it is also clear that $r_s(B_{n,k}) = \binom{n-k}{s}$. Thus by Theorem 1,

$$\begin{aligned} P_{n,k}(x) &= \sum_{s=0}^{n} P_{n,k,s} x^s \\ &= \sum_{s=0}^{n} h_{s,n}(B_{n,k}) x^s \\ &= \sum_{s=0}^{n} r_s(B_{n,k})(n-s)!(x-1)^s \\ &= \sum_{s=0}^{n} \binom{n-k}{s}(n-s)!(x-1)^s. \end{aligned} \tag{49}$$

Taking the coefficient of x^s on both sides, we have the following theorem due to Rakotondrajao [49].

Theorem 9. *For $n \geq 1$, $0 \leq k \leq n$, and $s \geq 0$,*

$$P_{n,k,s} = \sum_{t=s}^{n-k} (-1)^{t-s}(n-t)! \binom{t}{s} \binom{n-k}{t}. \tag{50}$$

Now Rakotondrajao [49] and Liese and Remmel [41] proved that (50) implies a number of simple relations among the coefficients $P_{n,k,s}$. For example, the following hold.

1. For $n \geq 2$, $0 \leq k < n$, and $s \geq 1$,

$$P_{n,k,s} = (n-s-1)P_{n-1,k,s} + (s+1)P_{n-1,k,s+1} + P_{n-1,k,s-1}. \tag{51}$$

2. For $n \geq 1$, $0 \leq k < n$, and $s \geq 0$,

$$P_{n,k,s} = \binom{n-k}{s} P_{n-s,k,0}. \tag{52}$$

3. For $n \geq 2$ and $0 \leq k < n$,

$$P_{n,k,0} = (n-1)P_{n-1,k,0} + (n-1-k)P_{n-2,k,0}. \tag{53}$$

4. For $n \geq 2$ and $0 < k < n$,

$$P_{n,k,0} = kP_{n-1,k-1,0} + (n-k)P_{n-1,k,0}. \tag{54}$$

5. For $n \geq 1$ and $0 \leq k < n$,

$$P_{n,k,0} = k! \sum_{r=0}^{k} \binom{k}{r} \binom{n-k}{k-r} P_{n-k,k-r,0}. \tag{55}$$

6. For $n \geq 2$ and $0 \leq k < n$,

$$P_{n,k,0} = P_{n,k+1,1} + kP_{n-1,k,0}. \tag{56}$$

7. For $n \geq 2$ and $0 < k < n$,

$$P_{n,k,0} = P_{n,k-1,0} + P_{n-1,k-1,0}. \tag{57}$$

8. For $n \geq 1$ and $0 \leq k < n$,

$$P_{n,k,0} = \sum_{r=0}^{k} \binom{k}{r} D_{n-k+r}. \tag{58}$$

Here Rakotondrajao [49] prove (51), (52), and (53) and Liese and Remmel [41] proved (54), (55), (56) and (58). Note (57) is just the usual recursion for the number of derangements. Liese and Remmel [41] showed that these recursions have simple combinatorial proofs when we think of $P_{n,k,s}$ as counting $H_s(B_{n,k})$. We will give a couple of examples of such proofs and leave the proofs of the rest of recursions as exercises.

Theorem 10. *For $n \geq 2$, $0 \leq k < n$, and $s \geq 1$,*

$$P_{n,k,s} = (n-s-1)P_{n-1,k,s} + (s+1)P_{n-1,k,s+1} + P_{n-1,k,s-1}. \tag{59}$$

Proof. We shall consider two different ways to insert $n+1$ into a permutation $\sigma = \sigma_1 \ldots \sigma_{n-1} \in S_{n-1}$. In our first type of insertion process, when we insert n into the i-th position, we obtain the permutation

$$I_i^1(\sigma) = \sigma_1 \ldots \sigma_{i-1}\ n\ \sigma_{i+1} \ldots \sigma_{n-1}\sigma_i$$

if $i \leq n-1$ and

$$I_n^1(\sigma) = \sigma_1 \ldots \sigma_{n-1}\ n$$

if $i = n$. Thus in our first insertion process, when we *insert n into the i-th position for $i < n$*, n replaces σ_i and bumps σ_i to the end. Our second insertion process is more standard. That is, when we insert n into the i-th position, we obtain the permutation

$$I_i^2(\sigma) = \sigma_1 \ldots \sigma_{i-1}\ n\ \sigma_i \ldots \sigma_{n-1}$$

if $i < n$ and

$$I_{n+1}^2(\sigma) = \sigma_1 \ldots \sigma_{n-1}\ n$$

if $i = n$. Thus in our second insertion process, when we *insert n into the i-th position for $i < n$*, n is inserted immediately in front of σ_i.

Given our first insertion process, it is easy to see that insertion of n can cause one extra k-excedance if we insert n into position $n-k$, we can decrease the excedances by 1 if we insert n into position i where $\sigma_i - i = k$, and we leave the number of k-excedances fixed otherwise. Thus it easily follows that

$$P_{n,k,s} = (n-s-1)P_{n-1,k,s} + (s+1)P_{n-1,k,s+1} + P_{n-1,k,s-1}$$

for all $n \geq 2$, $k \geq 0$, and $s \geq 1$.

Similarly we can use the second insertion process to do the same thing where we interpret $P_{n,k,s}$ as the number of permutations with s k-descents when $k \geq 1$. That is, we can create an extra k-descent if we insert n immediately in front of $n-k$, we can lose a k-descent if we insert n immediately in front of σ_{i+1} where $\sigma_i - \sigma_{i+1} = k$, and we keep the number of k-descents fixed otherwise. □

Theorem 11. *For $n \geq 1$, $0 \leq k < n$, and $s \geq 0$,*

$$P_{n,k,s} = \binom{n-k}{s} P_{n-s,k,0}. \tag{60}$$

Proof. The easiest way to give a combinatorial proof of this theorem is to think of the rook board $B_{n,k}$. Since $P_{n,k,s}$ counts the number of permutations of S_n having s k-excedances, it follows that $P_{n,k,s}$ is the number of placements of n non-attacking rooks in $[n] \times [n]$ that intersect $B_{n,k}$ in exactly s squares. If we then remove the rows and columns of those rooks which hit the diagonal connecting $(1, 1+k)$ and $(n-k, n)$, it easy to see that we will reduce ourselves to a permutation that is counted by $P_{n-s,k,0}$. This process is pictured in Figure 11. $\square$

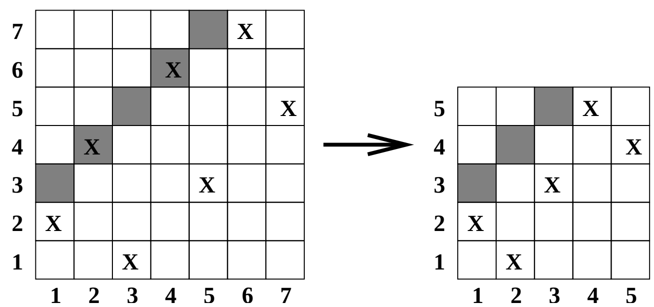

Figure 11: Removing rooks that lie in $B_{n,k}$.

Theorem 12. *For $n \geq 2$ and $0 \leq k < n$,*

$$P_{n,k,0} = (n-1)P_{n-1,k,0} + (n-1-k)P_{n-2,k,0}. \tag{61}$$

Proof. Think of reversing the first insertion process described in Theorem 10. That is, given a permutation $\sigma = \sigma_1 \dots \sigma_n \in S_n$, we let

$$\bar{\sigma} = \sigma_1 \dots \sigma_{i-1}\ \sigma_n\ \sigma_{i+1} \dots \sigma_{n-1}$$

if $\sigma_i = n$ for $i < n$ and let

$$\bar{\sigma} = \sigma_1 \dots \sigma_{n-1}$$

if $\sigma_n = n$. There are now two cases.

Case 1 $\bar{\sigma}$ had no k-excedances.
Note $\bar{\sigma}$ will have no k-excedances if either $\sigma_n = n$ or $\sigma_i = n$ with $i < n$ and $\sigma_n \neq i + k$. Vice versa, given a $\tau \in S_{n-1}$ with no k-excedances, we can use the first insertion process and insert n into τ in every position except $n - k$ to obtain a permutation in S_n with no k-excedances. Thus there are a $(n-1)P_{n-1,k,0}$ permutations in case 1.

Case 2 $\bar{\sigma}$ has 1 k-excedance.
This happens only if there is an $i \leq n-1-k$ such that $\sigma_i = n$ and $\sigma_n = i + k$. However in this case, we let $\tilde{\sigma}$ denote the permutation of S_{n-2} which results by removing the i-column and $i+k$-th row from the rook placement that corresponds to $\bar{\sigma}$ in $B_{n-1,k}$ as we did in the proof of

Theorem 11. It is easy to see that $\tilde{\sigma}$ has no k-excedances. Vice versa, it is easy to see that we can start with any permutation $\tilde{\tau} \in S_{n-2}$ that has no k-excedances and $i \leq n-1-k$ and obtain a permutation $\bar{\tau} \in S_{n-1}$ so that its corresponding rook placement has a rook in column i and row $i+k$ and reduces to the rook placement corresponding to $\tilde{\tau}$ when we remove the i-th and $i+k$-th row. Then we can obtain a $\sigma \in S_n$ with no k-excedances by inserting n into position i in $\bar{\tau}$ and moving $i+k$ to the end. Thus there are $(n-1-k)P_{n-2,k,0}$ permutations in Case 2. □

We note that Theorem 12 is a nice generalization of the basic theorem for the number of derangements. That is, $P_{n,0,0} = D_n$ where D_n is the number of derangements of S_n and (61) reduces to

$$D_n = (n-1)D_{n-1} + (n-1)D_{n-2}. \tag{62}$$

In fact, a moments thought will convince one that our proof reduces to the usual proof of this recursion in this case. That is, the permutations in Case 2 are the derangements where n in in 2-cycle and the permutations in Case 1 are derangements when n is not in a 2-cycle.

Theorem 13. *For $n \geq 2$ and $0 < k < n$,*

$$P_{n,k,0} = kP_{n-1,k-1,0} + (n-k)P_{n-1,k,0}. \tag{63}$$

Proof. This recursion is simply the result of classifying the rook placements corresponding to permutations $\sigma \in S_n$ with no k-excedances by the position of the rook in row 1. That is, there are two cases.

Case 1. The rook in the bottom row lies in column i where $i \leq n-k$.
In this case, we consider the rook placement that results by removing column i and row $i+k$. This will result in a placement of $n-2$ non-attacking rooks such no rook lies in $B_{n-1,k}$. We need to add a rook in bottom row and we do this so that the resulting rook placement is non-attacking. This will leave us with a rook placement of $n-1$ rooks in $[n-1] \times [n-1]$ that does not intersect $B_{n-1,k}$. Hence there are $(n-k)P_{n-1,k,0}$ such placements. This type of reduction is pictured at the top of Figure 12.

Case 2. The rook in the bottom row lies in column i where $i > n-k$.
In this case, removing row 1 and column i will result in a rook placement that does not intersect $B_{n-1,k-1}$. Thus there are $kP_{n-1,k-1,0}$ such rook placements. This type of reduction is pictured at the bottom of Figure 12

□

Theorem 14. *For $n \geq 1$ and $0 \leq k < n$,*

$$P_{n,k,0} = k! \sum_{r=0}^{k} \binom{k}{r} \binom{n-k}{k-r} P_{n-k,k-r,0}. \tag{64}$$

Proof. $P_{n,k,0}$ the number of placements of n non-attacking rooks on the $n \times n$ grid that never hit $B_{n,k}$. Consider the lightly shaded cells in the lower right hand corner of the board shown in Figure 13. There can be anywhere from 0 to k rooks placed in this square area. Suppose that we choose to place r rooks in this area. First we choose the r rows which will contain the

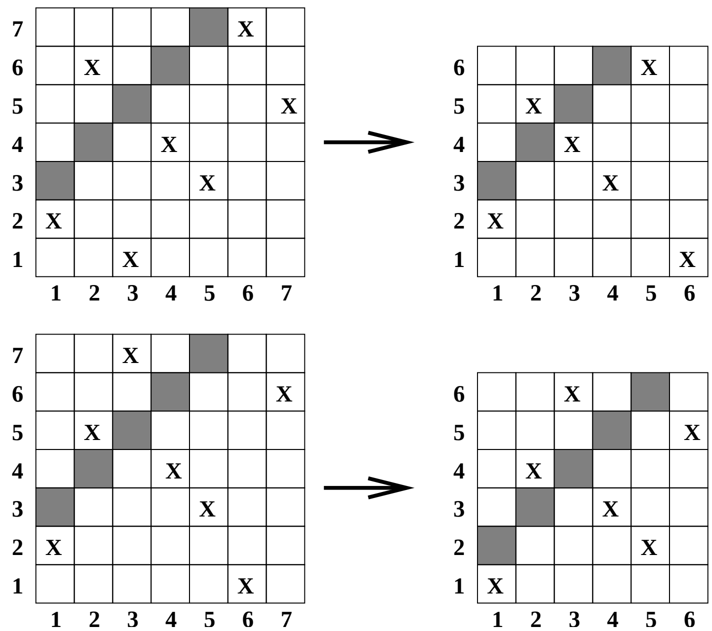

Figure 12: Reducing rook placements corresponding to $P_{n,k,0}$ by the position of the rook the first row.

rooks in this area in $\binom{k}{r}$ ways. Since there must be k rooks in the last k columns, there must be $k-r$ rooks in the rectangular region above the lightly shaded cells and we can choose the $k-r$ rows which will contain these rooks in $\binom{n-k}{k-r}$. Having picked the k rows that contain the rooks in the last k columns, there are $k!$ ways to place the rooks in the last k columns. Thus there $k!\binom{k}{r}\binom{n-k}{k-r}$ ways to pick a placement P of k non-attacking rooks in the last k columns so that r rooks fall in the lightly shaded area. Finally, we must count the number of ways to extend such a placement P to a placement Q of n non-attacking rooks in $[n] \times [n]$ so that no rook lies in $B_{n,k}$. If one thinks of removing the rows and columns of the rooks in P, it is easy to see that we are left with the board $B_{n-k,k-r}$ so that are $P_{n-k,k-r,0}$ ways to pick Q. Summing over all possible values of r yields the result. $\square$

Rakotondrajao [49] also derived the following generating functions for the $P_{n,k,s}$.

Theorem 15.

$$P_k(x,t) = \sum_{n\geq 0}\sum_{s=0}^{n} P_{n+k,k,s}\frac{x^s t^n}{n!} = \frac{k!e^{t(x-1)}}{(1-t)^{k+1}} \tag{65}$$

Proof. We need only show that the coefficient of $x^s t^{n-k}$ in $\dfrac{(n-k)!k!e^{t(x-1)}}{(1-t)^{k+1}}$ is in fact equal to the formula for $P_{n,k,s}$ that we previously demonstrated in (50). Using the fact that $(1-t)^{-(k+1)} =$

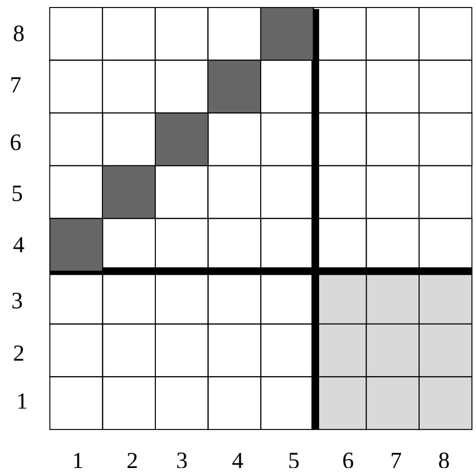

Figure 13: The board, $B_{8,3}$, with some lightly shaded cells.

$\sum_{m\geq 0}\binom{m+k}{k}t^m$, we find that

$$\begin{aligned}\frac{(n-k)!k!e^{t(x-1)}}{(1-t)^{k+1}} &= (n-k)!k!\left(\sum_{m\geq 0}\frac{(x-1)^m t^m}{m!}\right)\left(\sum_{m\geq 0}\binom{m+k}{k}t^m\right)\\ &= (n-k)!k!\left(\sum_{m\geq 0}\frac{t^m}{m!}\sum_{j=0}^{m}\binom{m}{j}(-1)^{m-j}x^j\right)\left(\sum_{m\geq 0}\binom{m+k}{k}t^m\right).\end{aligned}$$

Now taking the coefficient of t^{n-k} gives

$$(n-k)!k!\sum_{i=0}^{n-k}\sum_{j=0}^{i}\binom{i}{j}\frac{(-1)^{i-j}}{i!}x^j\binom{n-i}{k}.$$

Finally, we take the coefficient of x^s and arrive at

$$\begin{aligned}P_{n,k,s} &= (n-k)!k!\sum_{i=s}^{n-k}\binom{i}{s}\frac{(-1)^{i-s}}{i!}\binom{n-i}{k}\\ &= \sum_{i=s}^{n-k}(-1)^{i-s}\binom{i}{s}\frac{(n-i)!(n-k)!}{(n-i-k)!i!}\\ &= \sum_{i=s}^{n-k}(-1)^{i-s}(n-i)!\binom{i}{s}\binom{n-k}{i}\end{aligned}$$

which matches formula (50). □

Notice that (65) is a generalization of the well known generating function for derangements. Namely that

$$\sum_{n\geq 0}\sum_{s=0}^{n}P_{n,0,s}\frac{x^s t^n}{n!} = \sum_{n\geq 0}\sum_{s=0}^{n}D_s(n)\frac{x^s t^n}{n!} = \frac{e^{t(x-1)}}{1-t},$$

where $D_s(n)$ is the number of permutations on n elements having exactly s fixed points.

We can now generalize this generating function even further and consider

$$P(x,t,z) := \sum_{k\geq 0}\sum_{n\geq 0}\sum_{s=0}^{n} P_{n+k,k,s}\frac{x^s t^n z^k}{n!k!}. \tag{66}$$

Theorem 16.

$$P(x,t,z) = \frac{e^{t(x-1)}}{1-t-z} \tag{67}$$

Proof. Simply multiplying both sides of (65) by z^k and summing over all $k \geq 0$ gives this result. $\square$

Notice again that when $z = 0$, this reduces to the generating function for derangements.

Example 3: (X,Y)**-descents.**

Given subsets $X, Y \subseteq \mathbb{N}$ and a permutation $\sigma \in S_n$, let

$$\begin{aligned} Des_{X,Y}(\sigma) &= \{i : \sigma_i > \sigma_{i+1} \ \& \ \sigma_i \in X \ \& \ \sigma_{i+1} \in Y\}, \text{ and} \\ des_{X,Y}(\sigma) &= |Des_{X,Y}(\sigma)|. \end{aligned}$$

If $i \in Des_{X,Y}(\sigma)$, then we call the pair (σ_i, σ_{i+1}) an (X,Y)-*descent*.

For example, if $X = \{2,3,5\}, Y = \{1,3,4\}$, and $\sigma = 54213$, then $Des_{X,Y}(\sigma) = \{1,3\}$ and $des_{X,Y}(\sigma) = 2$.

For fixed n we define the polynomial

$$P_n^{X,Y}(x) = \sum_{s\geq 0} P_{n,s}^{X,Y} x^s := \sum_{\sigma\in S_n} x^{des_{X,Y}(\sigma)}. \tag{68}$$

Thus the coefficient $P_{n,s}^{X,Y}$ is the number of $\sigma \in S_n$ with exactly s (X,Y)-descents.

Kitaev and Remmel [39, 40] were the first to consider the distribution of $des_{X,Y}(\sigma)$ for certain X and Y when the studied descents according to the equivalence class mod k of either the top or bottom of a descent pair. For any set $X \subseteq \{1,2,3,\ldots\}$, they defined

- $\overleftarrow{Des}_X(\sigma) = \{i : \sigma_i > \sigma_{i+1} \ \& \ \sigma_i \in X\}$ and $\overleftarrow{des}_X(\sigma) = |\overleftarrow{Des}_X(\sigma)|$, and
- $\overrightarrow{Des}_X(\sigma) = \{i : \sigma_i > \sigma_{i+1} \ \& \ \sigma_{i+1} \in X\}$ and $\overrightarrow{des}_X(\sigma) = |\overrightarrow{Des}_X(\sigma)|$.

It is easy to see that $\overleftarrow{Des}_X(\sigma) = Des_{X,\mathbb{N}}(\sigma)$ and $\overrightarrow{Des}_X(\sigma) = Des_{\mathbb{N},X}(\sigma)$. In [39], Kitaev and Remmel studied polynomials such as

$$\begin{aligned} R_n(x) &= \sum_{k\geq 0} R_{k,n}x^k := \sum_{\sigma\in S_n} x^{\overleftarrow{des}_{\mathbb{E}}(\sigma)}, \text{ and} \\ Q_n(x) &= \sum_{k\geq 0} Q_{k,n}x^k := \sum_{\sigma\in S_n} x^{\overrightarrow{des}_{\mathbb{E}}(\sigma)}, \end{aligned}$$

where $\mathbb{E}$ is the set of positive even integers. In these cases, they found surprisingly simple formulas for the coefficients. For example, they showed that

$$R_{2n,k} = \binom{n}{k}^2 (n!)^2. \tag{69}$$

Kitaev and Remmel originally proved (69) by recursion. However, in this example, we will show how we can derive such a formula from a rook theory point of view following ideas of Hall and Remmel [36]. That is, it is not difficult to use Foata first fundamental transformation as described in example 2 to reduce the problem of computing $P_{n,s}^{X,Y}$ to that a computing a hit number for a given board.

Given set $X, Y, Z \subseteq \mathbb{P}$ and $\sigma = \sigma_1 \dots \sigma_n \in S_n$, let

$$\begin{aligned} Des_{X,Y,Z}(\sigma) &= \{i : \sigma_i > \sigma_{i+1}, \sigma_i \in X, \sigma_{i+1} \in Y, \ \& \ \sigma_i - \sigma_{i+1} \in S\} \text{ and} && (70)\\ des_{X,Y,Z}(\sigma) &= |Des_{X,Y,Z}(\sigma)|. && (71) \end{aligned}$$

Let $\mathbb{O}$ denote the set of odd numbers in $\mathbb{P}$. Now suppose we want to compute

$$\sum_{\sigma \in S_n} x^{des_{X,Y,Z}(\sigma)}$$

where $X = \mathbb{E}$, $Y = \mathbb{O}$, and and $Z = \{1, 3\}$. For $n = 8$, the board B_8^U consists of the squares $(i, j) \in [8] \times [8]$ such that $i \in \mathbb{E}, j \in \mathbb{O}$, and $i - j = 1$ or 3. We have pictured this board as the shaded squares in Figure 14. Now consider the placement, shown in Figure 15, of eight non-

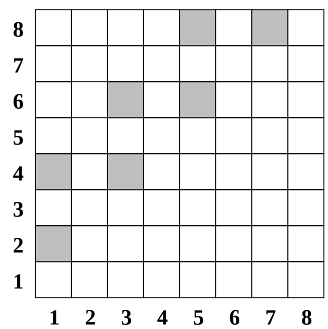

Figure 14: The board B_8^U.

attacking rooks (marked by X's) on the 8×8 board so that two rooks lie on B_8^U. This placement corresponds to the permutation $\omega = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 1 & 5 & 7 & 6 & 2 & 3 & 8 \end{pmatrix}$, with the rooks placed on B_8^U corresponding to the excedances $\begin{matrix} 1 \\ 4 \end{matrix}$ and $\begin{matrix} 5 \\ 6 \end{matrix}$ .Then Foata's first fundamental transformation maps ω to the permutation $\sigma = \Phi(\omega) = 74126538$ with exactly two X, Y, Z-descents: 41 and 65. Thus

$$\sum_{\sigma \in S_8} x^{des_{X,Y,Z}(\sigma)} = \sum_{k=0}^{8} H_{k,8}(B_8^U).$$

Now consider the the case where $X = \mathbb{E}$ and $Y = \mathbb{N}$ so that $des_{X,Y}(\sigma) = des_{\mathbb{E}}(\sigma)$. Using Foata's first fundamental transformation, it is easy to see that a descent $\sigma_i > \sigma_{i+1}$ where $\sigma_i \in \mathbb{E}$

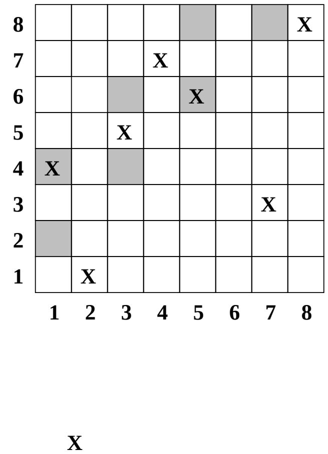

Figure 15: A placement of rooks on the 8×8 board.

corresponds to an excedance pair (σ_{i+1}, σ_i) in $\Phi^1(\sigma)$ where σ_i is even. Thus in this case, we should study hit polynomials for the board $B_n^{\mathbb{E},\mathbb{N}}$ which consists of squares of the form $(i, 2k)$ where $i < 2k$. For example, the board $B_8^{\mathbb{E},\mathbb{N}}$ corresponding to even descents of permutations $\sigma \in S_8$ is shown in Figure 16.

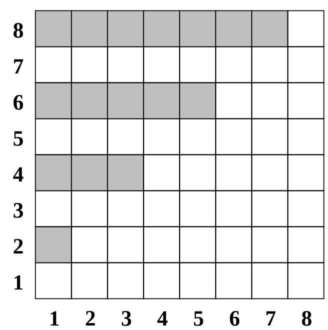

Figure 16: The board $B_8^{\mathbb{E},\mathbb{N}}$.

Now it is easy to see that permuting the rows of a board does not change the hit numbers. Similarly, permuting the columns of rook board does not change the hit numbers. Thus the board $B_8^{\mathbb{E},\mathbb{N}}$ has the same hit numbers as the Ferrers board is shown in Figure 17.

Thus, in general, $B_{2n}^{\mathbb{E},\mathbb{N}}$ has the same hit numbers as $F(0,1,1,2,2,\ldots,n-1,n-1,n)$ relative to $[2n] \times [2n]$ and $B_{2n+1}^{\mathbb{E},\mathbb{N}}$ has the same hit numbers as $F(0,0,1,1,2,2,\ldots,n-1,n-1,n)F(0,0,1,1,2,2,\ldots,n-1,n-1,n)$ relative to $[2n+1] \times [2n+1]$. However it is easy to see from Theorem 6 that $F(0,0,1,1,2,2,\ldots,n-1,n-1,n)$ is rook equivalent to $f(0^n, n^n)$. That is, it is easy to see that $\mathcal{S}(0,0,1,1,2,2,\ldots,n-1,n-1,n)$ and $\mathcal{S}(0^n,n^n)$ are both rearrangements of $0^2(-1)^2\cdots(-n)^2$. Similarly, it is easy to check that $F(0,0,1,1,2,2,\ldots,n-1,n-1,n)$ is rook equivalent of $F(0^{n+1},n^n)$. It thus follows that

$$P_{2n,s}^{\mathbb{E},\mathbb{N}} = H_{s,2n}(F(0^n,n^n)) \text{ and } \tag{72}$$

$$P_{2n+s,s}^{\mathbb{E},\mathbb{N}} = H_{s,2n+1}(F(0^{n+1},n^n)). \tag{73}$$

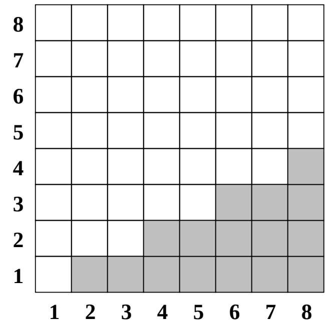

Figure 17: The Ferrers board B corresponding to $B_8^{\mathbb{E},\mathbb{N}}$.

But it is easy to see that

$$H_{s,2n}(F(0^n,n^n)) = (n!)^2\binom{n}{s}^2. \tag{74}$$

That is, suppose that we want to place s rooks in $B = F(0^n,n^n)$. There are $\binom{n}{s}$ ways to pick the s rows in which to place those rooks in B and $\binom{n}{s}$ ways to pick the s columns in which to place those rooks in B. Once we have picked the rows and columns for the s rooks in B, there are $s!$ ways to place those rooks. Thus there are $\binom{n}{s}^2 s!$ ways to place s rooks in B. Next we $n-s$ columns empty columns in the last n columns of $[2n]\times[2n]$. Such rooks must lie in rows $n+1,\ldots,2n$ so that there there $n(n-1)\ldots(n-(n-s)+1$ ways to place such rooks. Finally having chosen a placement of n rooks in the last n columns, there will be $n!$ ways to place the rooks in the first n columns. Thus

$$H_{s,2n}(F(0^n,n^n)) = \binom{n}{s}^2 s!(n)\downarrow_{n-s} n! = (n!)^2\binom{n}{s}^2.$$

Similarly, suppose that we want to place s rooks in $B' = F(0^{n+1},n^n)$. Then as before, there are $\binom{n}{s}^2 s!$ ways to place s rooks in B'. Next we $n-s$ columns empty columns in the last n columns of $[2n+1]\times[2n+1]$. Such rooks must lie in rows $n+1,\ldots,2n+1$ so that there there $(n+1)n\ldots(n+1-(n-s)+1 = (n+1)\downarrow_{n-s}$ ways to place such rooks. Finally having chosen a placement of n rooks in the last n columns, there will be $(n+1)!$ ways to place the rooks in the first $n+1$ columns. Thus

$$H_{s,2n+1}(F(0^{n+1},n^n)) = \binom{n}{s}^2 s!(n+1)\downarrow_{n-s}(n+1)! = \frac{1}{s+1}((n+1)!)^2\binom{n}{s}^2.$$

It should be clear from the two examples above that it is easy to compute the hit numbers for rectangular boards. Thus it is natural to ask whether there are other sets X such that the board corresponding to X-descents is rook equivalent to a rectangular board. In fact, Hall and Remmel [36] showed that S of the form $X = \{u+2,u+4,u+6,\ldots,u+2m\})$ covers all possibilities. For example, consider the 2×3 rectangular board B shown in Figure 18. We place this board in the lower right corner of an 8×8 board, as shown in Figure 19.

Note that for any set X the board associated to X-descents has distinct rows, since the row corresponding to $i\in X$ has length $i-1$. Rearranging the elements of $s(B)$ in weakly decreasing order gives the unique board B' with distinct rows that is rook-equivalent to B. In our example, we compute $s(B) = (0,-1,-2,-3,-4,-3,-4,-5)$. Thus B is rook-equivalent to

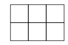

Figure 18: The 2×3 board B.

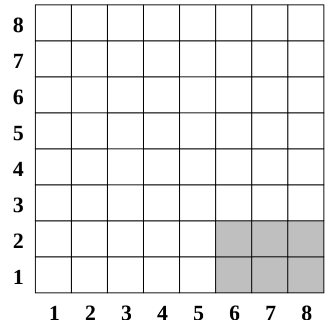

Figure 19: The board B placed in an 8×8 board.

the board B' with structure vector $s(B') = (0, -1, -2, -3, -3, -4, -4, -5)$ and height vector $h(B') = (0, 0, 0, 0, 1, 1, 2, 2)$. B' is the board shown in Figure 20. Finally, to get the board

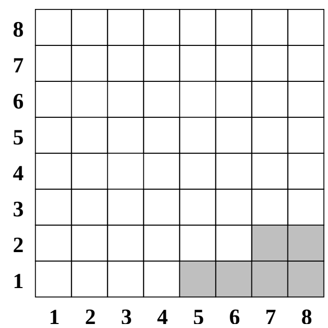

Figure 20: The board B', rook-equivalent to B.

for X-descents, we take the mirror image, and shift the rows upwards so that a row of length $i - 1$ is in position i, as shown in Figure 21. Thus in our example, $X = \{3, 5\}$. In general, Hall and Remmel observed that if one starts with a rectangular $a \times b$ board ($a \leq b$), then the corresponding set is $X = \{u + 2, u + 4, u + 6, \ldots, u + 2m\}$, where $m = a$ and $u = b - a$. For $n = 2m + u + v$, we get

$$
\begin{aligned}
P^X_{n,s} &= \binom{a}{s}\binom{b}{s} s! \cdot \binom{n-a}{b-s} (b-s)! \cdot (n-b)! \\
&= \binom{m}{s}\binom{m+u}{s} s! \cdot \binom{m+u+v}{m+u-s} (m+u-s)! \cdot (m+v)! \\
&= \binom{m}{s}\binom{m+u+v}{v+s} (m+u)!(m+v)!.
\end{aligned}
$$

In many cases, one can use the fact that each Ferrers board is rook equivalent to a unique increasing Ferrers board to show that the problem counting (X, Y)-descents can be reduced the problem of counting X'-descents for an appropriate X'. That is, Hall and Remmel proved the following.

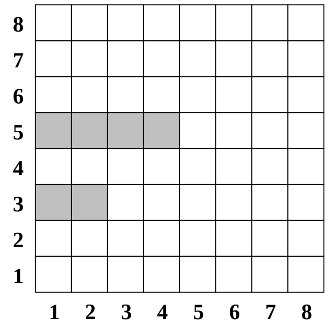

Figure 21: The board B_8^X.

Proposition 1. *Given subsets $X, Y \subseteq \mathbb{N}$, let B be the Ferrers board corresponding to the potential descent pairs (i, j), where $i \in X_n$ and $j \in Y_n$. Let B' be the unique Ferrers board rook-equivalent to B that has distinct rows. Let $X' \subseteq [n]$ be the unique subset whose corresponding board (once empty rows and columns are deleted, and taking the mirror image) is B'. Then*

$$P_{n,s}^{X,Y} = P_{n,s}^{X'}.$$

Proof. By the previous discussion, we have $P_{n,s}^{X,Y} = H_{s,n}(B)$ and $P_{n,s}^{X'} = H_{s,n}(B')$. But B and B' are rook-equivalent, and thus $H_{s,n}(B) = H_{s,n}(B')$ for all s. □

For example, suppose that $X = \{2, 3, 5, 7, 8\}, Y = \{1, 2, 4, 5, 6\}$, and $n = 8$, so that the potential descent pairs are $21, 31, 32, 51, 52, 54, 71, 72, 74, 75, 76, 81, 82, 84, 85,$ and 86. Then $P_{8,s}^{X,Y} = H_{s,8}(B_8^{X,Y})$, where $B_8^{X,Y}$ is the board shown on the left in Figure 22. This board is clearly hit-equivalent to the Ferrers board $F(0, 0, 0, 2, 2, 3, 4, 5)$ shown on the right in Figure 22. It is easy to check that the unique increasing Ferrers board B' which rook equivalent to $F(0, 0, 0, 2, 2, 3, 4, 5)$ is $F(0, 0, 0, 1, 2, 3, 4, 6)$ which is shown on the left in Figure 23. It is easy to see that B' is hit-equivalent to the board for $X' = \{2, 3, 4, 5, 7\}$ which is shown on the right in Figure 23. Thus $P_{8,s}^{X,Y} = P_{8,s}^{X'}$ for all s.

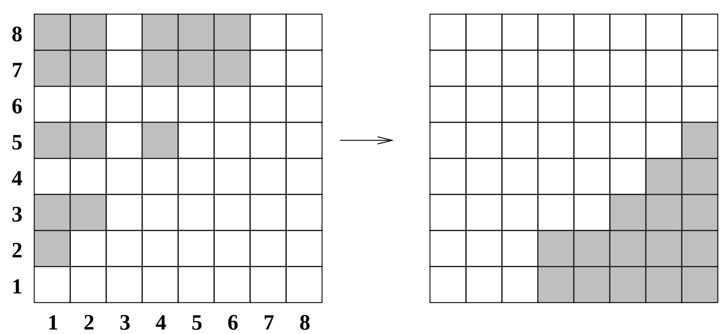

Figure 22: The board $B_8^{X,Y}$.

Hall and Remmel [36] were able to find general formulas for $P_{n,s}^{X,Y}$. In fact, they gave direct combinatorial proofs of a pair of formulas for $P_{n,s}^{X,Y}$. Their proofs did not use rook theory

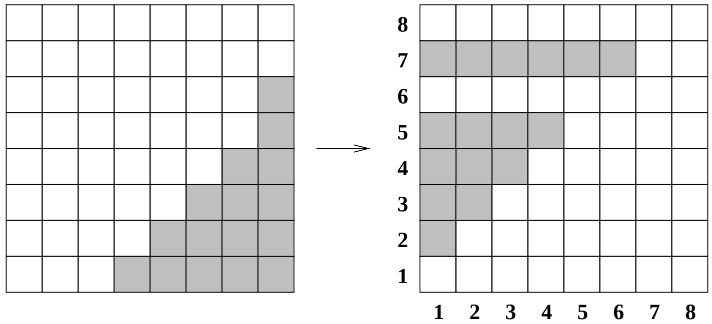

Figure 23: The board B', rook-equivalent to $B_8^{X,Y}$.

methods, but we state them for completeness. First of all, for any set $A \subseteq \mathbb{N}$, let

$$\begin{aligned} A_n &= A \cap [n], \text{ and} \\ A_n^c &= (A^c)_n = [n] - A. \end{aligned}$$

Then we have

Theorem 17.

$$P_{n,s}^{X,Y} = |X_n^c|! \sum_{r=0}^{s} (-1)^{s-r} \binom{|X_n^c|+r}{r} \binom{n+1}{s-r} \prod_{x \in X_n} (1 + r + \alpha_{X,n,x} + \beta_{Y,n,x}), \tag{75}$$

Theorem 18.

$$P_{n,s}^{X,Y} = |X_n^c|! \sum_{r=0}^{|X_n|-s} (-1)^{|X_n|-s-r} \binom{|X_n^c|+r}{r} \binom{n+1}{|X_n|-s-r} \prod_{x \in X_n} (r + \beta_{X,n,x} - \beta_{Y,n,x}), \tag{76}$$

where for any set A and any $j, 1 \le j \le n$, we define

$$\begin{aligned} \alpha_{A,n,j} &= |A^c \cap \{j+1, j+2, \ldots, n\}| = |\{x : j < x \le n \;\&\; x \notin A\}|, \text{ and} \\ \beta_{A,n,j} &= |A^c \cap \{1, 2, \ldots, j-1\}| = |\{x : 1 \le x < j \;\&\; x \notin A\}|. \end{aligned}$$

Example 1. *Suppose $X = \{2,3,4,6,7,9\}, Y = \{1,4,8\}$, and $n = 6$. Thus $X_6 = \{2,3,4,6\}, X_6^c = \{1,5\}, Y_6 = \{1,4\}, Y_6^c = \{2,3,5,6\}$, and we have the following table of values of $\alpha_{X,6,x}, \beta_{Y,6,x}$, and $\beta_{X,6,x}$.*

x	2	3	4	6
$\alpha_{X,6,x}$	1	1	1	0
$\beta_{Y,6,x}$	0	1	2	3
$\beta_{X,6,x}$	1	1	1	2

Equation (75) gives

$$\begin{aligned} P_{6,2}^{X,Y} &= 2! \sum_{r=0}^{2} (-1)^{2-r} \binom{2+r}{r} \binom{7}{2-r} (2+r)(3+r)(4+r)(4+r) \\ &= 2\,(1 \cdot 21 \cdot 2 \cdot 3 \cdot 4 \cdot 4 - 3 \cdot 7 \cdot 3 \cdot 4 \cdot 5 \cdot 5 + 6 \cdot 1 \cdot 4 \cdot 5 \cdot 6 \cdot 6) \\ &= 2(2016 - 6300 + 4320) \\ &= 72. \end{aligned}$$

while (76) gives

$$\begin{aligned}
P_{6,2}^{X,Y} &= 2!\sum_{r=0}^{2}(-1)^{2-r}\binom{2+r}{r}\binom{7}{2-r}(1+r)(0+r)(-1+r)(-1+r)\\
&= 2\left(1\cdot 21\cdot 1\cdot 0\cdot(-1)\cdot(-1) - 3\cdot 7\cdot 2\cdot 1\cdot 0\cdot 0 + 6\cdot 1\cdot 3\cdot 2\cdot 1\cdot 1\right)\\
&= 2(0-0+36)\\
&= 72.
\end{aligned}$$

Example 4. Eulerian numbers and Simon Newcomb's problem.

Another way to identify rook placements with permutations is to start with a permutation π in one-line notation, then create another permutation $\beta(\pi)$ by viewing each left-to-right minima of π as the last element in a cycle of $\beta(\pi)$. For example, if $\pi = 361295784$, then $\beta(\pi) = (361)(2)(95784)$. In this example $P(\beta(\pi))$ consists of rooks on

$$\{(3,6),(6,1),(1,3),(2,2),(9,5),(5,7),(7,8),(8,4),(4,9)\}. \tag{77}$$

Note that the number of permutations with k cycles is hence equal to the number of permutations with k left-to-right minima. Now for the rook placement $P(\beta(\pi))$, a rook on (i,j) can be interpreted as meaning j immediately follows i in some cycle of β, and if $i > j$, this will happen iff π contains the descent $\cdots ij \cdots$. If we let St_n denote the "staircase board" consisting of squares $(i,j), 1 \le j < i \le n$, then rooks on B_n in $P(\beta(\pi))$ correspond to descents in π. Hence we have

$$H_{k,n}(St_n) = A_{k+1}(n), \tag{78}$$

where $A_j(n)$ is the jth "Eulerian number", i.e. the number of permutations in S_n with $j-1$ descents.

Identity (78) has a nice generalization to multiset permutations. Given $v \in \mathbb{N}^p$, define a map ζ from S_n to the set of multiset permutations $M(v_1, v_2, \ldots, v_p)$ of $\{1^{v_1}\cdots p^{v_p}\}$ by starting with $\pi \in S_n$ and replacing the smallest v_1 numbers by 1's, the next v_2 smallest numbers by 2's, etc. For example, if $\pi = 361295784$ and $v = (3,5,1)$, then $\zeta(\pi) = 121132222$. Note that with this v, rooks on squares $(2,1),(3,1),(3,2)$ no longer correspond to descents, and neither do rooks on $(5,4),(6,4),(6,5),\ldots,(8,7)$.

Let $N_k(v)$ denote the number of multiset permutations of elements of $M(v)$ with $k-1$ descents. The $N_k(v)$ are named after British astronomer Simon Newcomb who, while playing a card game called patience, posed the following problem: if we deal the cards of a 52 card-deck out one at a time, starting a new pile whenever the face value of the card is less than that of the previous card, in how many ways can we end up with exactly k piles? MacMahon noted this is equivalent to asking for a formula for $N_k(13,13,13,13)$, and studied the more general question of finding a formula for $N_k(v)$ [42]. Riordan [48] noted that since ζ is a 1 to $\prod_i v_i!$ map, if we let G_v denote the Ferrers board whose first v_1 columns are of height 0, next v_2 of height v_1, next v_3 of height v_1+v_2, $\ldots$, and last v_p of height $v_1+\ldots v_{p-1}$, it follows that

$$H_{k,n}(G_v) = \prod_i v_i! N_{k+1}(v). \tag{79}$$

Example 5: Problème des ménages.

The problème des ménages is a typical arrangement problem. In this case, the problem is that we start our with n husbands and wives $(H_1, W_1), \ldots (H_n, W_n)$ that we want to sit around a circular table in such a way that (i) husbands and wives alternate and (ii) no husband sits next to his wife. The question is to find the number A_n of seating arrangements that satisfy conditions (i) and (ii) up to circular rearrangements?

We solve this problem is several steps. First we seat the wives around the table in some order $V_1, \ldots, V_n$. Clearly there are $(n-1)!$ ways to do this up to circular rearrangements. For example, if $n = 8$, we have a seating arrangement as pictured in Figure 24. We let P_i be the place just to the right of V_i. Now if G_i is the husband of V_i for $i = 1, \ldots, n$, then our goal is to find the number of ways to place $G_1, \ldots, G_n$ into the place $P_1, \ldots, P_n$ so that G_1 is not in either P_1 or P_n and, for $i = 2, \ldots, n$, G_i is not in either P_{i-1} or P_i. Clearly, we can think of assignment of the husbands to the places a permutation $\sigma = \sigma_1 \ldots \sigma_n \in S_n$ such that $\sigma_1 \notin \{1, n\}$ and, for $i = 2, \ldots, n$, $\sigma_i \notin \{i-1, i\}$. The number of such permutations is the hit number $H_{0,n}(C_n)$ where C_n is board contained in $[n] \times [n]$ where

$$C_n = \{(1, n), (1, 1)\} \cup \{(i, i-1), (i, i) : i = 2, \ldots n\}.$$

For example, the board C_8 is pictured in Figure 25 on the left where the shaded squares correspond to the board C_8. Thus we know $A_n = (n-1)! H_{0,n}(C_n)$.

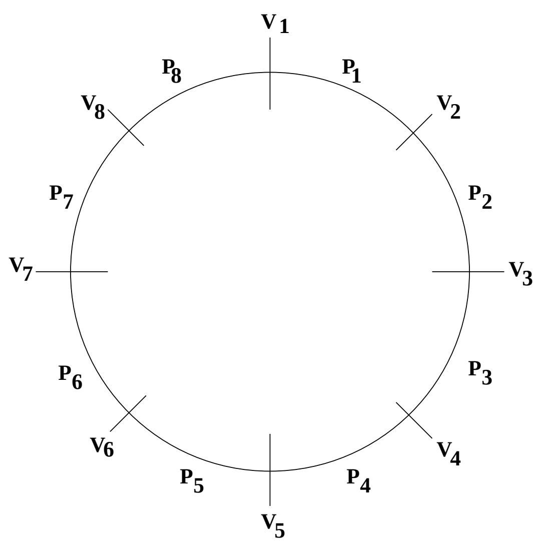

Figure 24: First step of the seating arrangement.

We can then use Theorem 1 to conclude that

$$H_{0,n}(C_n) = \sum_{k=0}^{n} r_k(C_n)(-1)^k (n-k)!. \tag{80}$$

Hence, we can obtain a formula for A_n if we can find a formula of $r_k(C_n)$ for $k = 0, \ldots, n$.

If one adds put a vertex in each of the squares of C_n and connect consecutive vertices by an edge, one obtain a cycle $\mathcal{C}_{2n}$. For example, in Figure 25, we see the cycle that corresponds

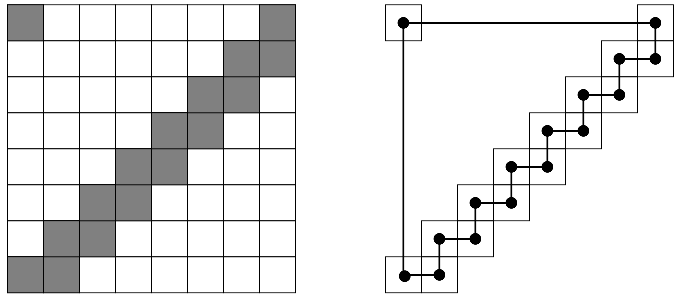

Figure 25: The board C_8.

to the board C_8 is cycle $\mathcal{C}_{16}$. It is then easy to see that $r_s(C_n)$ is just the number of ways of picking s vertices in the cycle $\mathcal{C}_{2n}$ so that no two elements are connected by an edge. We then have the following lemma that can be found in Stanley's book [50].

Lemma 19. *Let $\mathcal{C}_n$ denote the graph of an n-cycle. Let $f(n,s)$ be the number of ways a picking s vertices from $\mathcal{C}_n$ such that no two are connected by an edge. Then $f(n,s) = \frac{n}{n-s}\binom{n-s}{s}$.*

Proof. First, let $g(n,s)$ be the number of ways a picking s vertices from $\mathcal{C}_n$ such that no two are connected by an edge and then circling one of the remaining $n-s$ vertices. Clearly, $g(n,s) = (n-s)f(n,s)$. Thus we need only show that $g(n,s) = n\binom{n-s}{s}$. Imagine that the vertices of $\mathcal{C}_n$ are labeled consecutive with $1,2,\ldots,n$ as we traverse the circle in clockwise fashion.

To construct a configuration counted by $g(n,s)$, we first pick the circled point in n ways. Then we place $n-s-1$ unlabeled points around the circle. These will correspond to the points that are not chosen when we label the points. For example, suppose that $n=9$ and $s=3$ and we choose the circled point to be 4. Then $n-s-1 = 9-3-1 = 5$ so that we would have the configuration pictured in Figure 26.

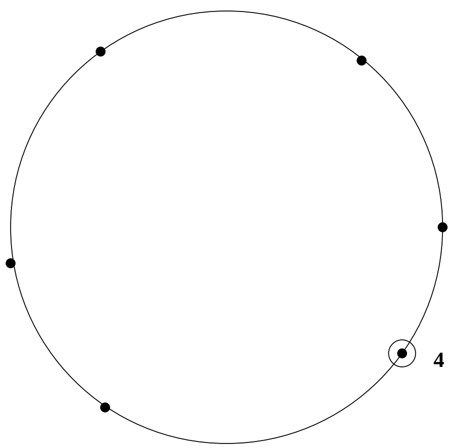

Figure 26: Placing the circled point plus $n-s-1$ other points.

At this point, we have $n-s$ spaces between consecutive vertices that we have placed on the circle so we choose s of them in $\binom{n-s}{s}$ ways. Once we have chosen the s points, then we know how to label the points on the circle. For example, if we continue with our example above, then in Figure 27 we have pictured such a labeling where we indicate the s chosen points with Xs.

□

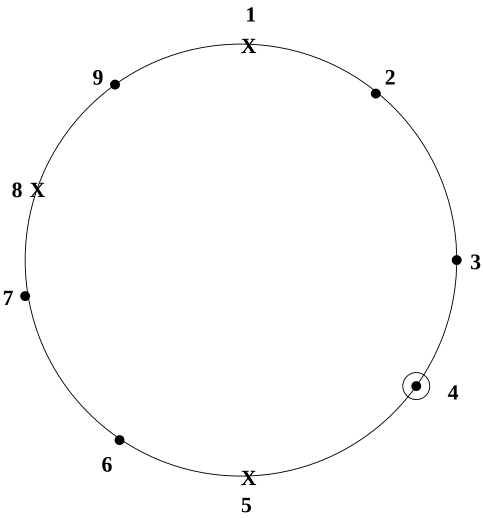

Figure 27: Completing a labeling.

Thus it follows that

$$A_n = (n-1)!H_{0,n}(C_n) = (n-1)!\sum_{k=0}^{n} \frac{2n}{2n-k}\binom{2n-k}{k}(-1)^k(n-k)!.$$

1.6 Rook numbers and Weyl Algebra

The Weyl Algebra is the algebra generated by two operators D and U subject to the relation $DU = UD + 1$. The simplest example of such an algebra to consider the polynomial ring $\mathbb{Q}[x]$ where we let D be the differentiation operator $\frac{d}{dx}$, U be the multiplication by x operator, and I be the identity operator on $\mathbb{Q}[x]$. Then

$$\begin{aligned} DU(x^n) &= D(x^{n+1}) = (n+1)x^n \text{ and} \\ (UD+I)(x^n) &= UD(x^n) + I(x^n) = U(nx^{n-1}) + x^n = nx^n + x^n = (n+1)x^n. \end{aligned}$$

For any alphabet A, we let A^* denote the set of all words $w = w_1 \ldots w_n$ such that $w_i \in A$ and we let A^n denote the set of all words in A^* of length n. For any word $w \in A^*$, we let $|w|$ denote the length of w. We let ϵ denote the empty word and declare $|\epsilon| = 0$.

The problem that we want to consider is to start out with a word $w \in \{D,U\}^*$ and find an expressions for the so-called normal form of w. That is, we want to find coefficients $c_{i,j}$ so that

$$w = \sum_{i,j} c_{i,j} U^i D^j. \tag{81}$$

We claim that the $c_{i,j}$'s are just rook numbers of a certain Ferrers board B_w associated with w. That is, we create a path $P(w)$ consisting of north and east steps of length 1 from w by having each D correspond to a north step and each U correspond to east step. Then B_w will be the Ferrers board whose left boundary corresponds the path $P(W)$. For example, $w = DDUDUDU$, then $P(W)$ and B_w are pictured in Figure 28.

Note that it is easy to see that if $u = U^j w D^k$, then $B_u = B_w$. Then Navon [43] proved the following Theorem.

Figure 28: The rook board associated with the word $w = DDUDUDU$.

Theorem 20. *Let $w \in \{D, U\}^*$ consists of n D's and m Us. Then*

$$w = \sum_{k=0}^{n} r_k(B_w) U^{m-k} D^{n-k} \tag{82}$$

Proof. The proof is a consequence of a simple recursion for the rook numbers of Ferrers boards. That is, suppose that c is outside corner square of a Ferrers board $B = F(b_1, \ldots, b_n)$. Then we let B/c denote the Ferrers board which results from B by removing cell c and we let $B/\bar{c}$ denote the Ferrers board which results from B by removing the row and column that contains c. For example, if $B = F(1, 3, 4, 4, 5)$ and c is the cell (3,2), then Figure 29 pictures both B/c and $B/\bar{c}$. Here we have put dots in cells besides c that we remove when we form $B/\bar{c}$.

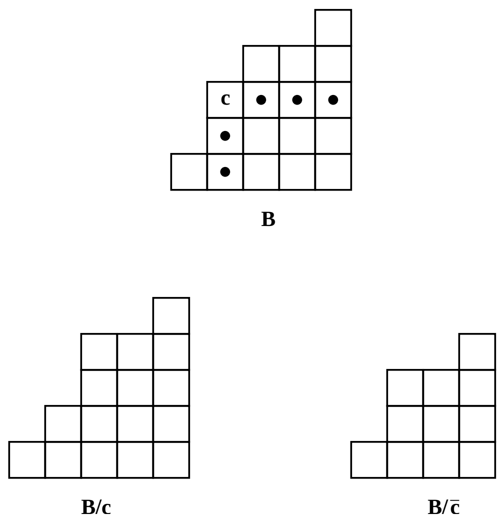

Figure 29: The boards B/c and $B/\bar{c}$.

By classifying the rook placements $P \in \mathcal{N}_k(B)$ according to whether P has a rook in cell c or not, it easy to see that for any $k \geq 1$,

$$r_k(B) = r_k(B/c) + r_{k-1}(B/\bar{c}). \tag{83}$$

That is, we can clearly partition the rook placements $P \in \mathcal{N}_k(B)$ into N_1, consisting of those rook placement that do not have a rook in cell c and N_2, consisting of those rooks that have a rook in cell c. Then it is easy to see that $|N_1| = r_k(B/c)$ and $|N_2| = r_{k-1}(B/\bar{c})$.

We can then prove (82) by induction on the number of cells in B_w. If B_w is empty, then w must be $U^m D^n$ so that (82) automatically holds in this case since by definition $r_0(B_w) = 1$

Now suppose that w is of the form D^sUv. Then clearly, the cell $c = (1, s)$ will be a corner square of B_w. Now

$$w = U^sD^tUv = U^sD^{t-1}(UD + 1)v = U^sD^{t-1}UDv + U^sD^{t-1}v.$$

But if $a = U^sD^{t-1}UDv$ and $b = U^sD^{t-1}v$, then it is easy to see that $B_a = B_w/c$ and $B_b = B_w/\bar{c}$. See Figure **??** for an example. But then by induction we have that

$$\begin{aligned} w &= U^sD^tUv = U^sD^{t-1}UDv + U^sD^{t-1}v \\ &= \sum_{k=0}^{n} r_k(B_a)U^{m-k}D^{n-k} + \sum_{k=0}^{n-1} r_k(B_b)U^{m-1-k}D^{n-1-k} \\ &= \sum_{k=0}^{n}(r(B_a) + r_{k-1}(B_b))U^{m-k}D^{n-k} \\ &= \sum_{k=0}^{n} r_w(B)U^{m-k}D^{n-k}. \end{aligned}$$

□

In fact, Theorem 20 can be used to show that classical combinatorial numbers such as the Stirling numbers of the second kind and the Lah numbers appear in normal form expansions in Weyl Algebra. That is, if $w = (UD)^{n-1}$, then B_w is the staircase board St_n. Thus by Theorem 20

$$\begin{aligned} (UD)^{n-1} &= \sum_{k=0}^{n-1} r_k(St_n)U^{n-1-k}D^{n-1-k} \\ &= \sum_{k=1}^{n} S_{n,k}U^{k-1}D^{k-1}. \end{aligned}$$

Similarly if $w = D^{n-1}U^n$, then B_w is the Laguerre board $\mathcal{L}_n$. Thus by Theorem 20

$$\begin{aligned} D^{n-1}U^n &= \sum_{k=0}^{n-1} r_k(\mathcal{L}_n)U^{n-1-k}D^{n-k} \\ &= \sum_{k=1}^{n} L_{n,k}U^{k-1}D^k. \end{aligned}$$

We should also observe that Theorem 20 and the theory of rook equivalences of Ferrers boards leads to non-trivial polynomial identities in the Weyl Algebra. That is, consider $w = DUDU^2D$ and $u = UD^2UDU$ Then $B_w = F(1, 2, 2)$ and $B_u = F(2, 3)$. But it easy to see that $F(1, 2, 2)$ and $F(2, 3)$ are rook equivalent so that by Theorem 20

$$DUDU^2D = \sum_{k=0}^{2} r_k(F(1, 2, 2))U^{3-k}D^{3-k} = \sum_{k=0}^{2} r_k(F(2, 3))U^{3-k}D^{3-k} = UD^2UDU.$$

Exercise: Show that in Weyl algebra, $DU^2D^2U^2D = UDUD^2UDU$.

1.7 Recursions for hit and fit numbers of Ferrers boards

In the next few sections, we shall develop some properties of hit and fit numbers of Ferrers boards. We start by looking at the recursion for rook numbers considered in the previous section. That is, suppose the c is outside corner square of a Ferrers board $B = F(b_1, \dots, b_n)$. Recall that B/c denotes the Ferrers board which results from B by removing cell c and $B/\bar{c}$ denote the Ferrers board which results from B by removing the row and column that contains c. Then we showed that

$$r_k(B) = r_k(B/c) + r_{k-1}(B/\bar{c}).$$

Similarly, suppose that $A = F(a_1, \dots, a_n)$ is a skyline board and $c = (a_i, i)$ is the top cell of a non-zero column. Then we let $A/\bar{\bar{c}}$ denote the skyline board $F(a_1, \dots, a_{i-1}, 0, a_{i+1}, \dots, a_n)$ be the board that results from A by removing all the cells in i-th column of A. Again by classifying the file placements $Q \in \mathcal{C}_k(A)$ according to whether they have a rook in cell c or not, it is easy to see that

$$f_k(A) = f_k(A/c) + f_{k-1}(A/\bar{\bar{c}}). \tag{84}$$

We can then use these two recursions and Theorems 1 and 2 to develop recursions for the hit numbers and fit numbers of Ferrers boards. That is, suppose that c is outside corner square of a Ferrers board $B = F(b_1, \dots, b_n) \subseteq [n] \times [n]$. Then since we remove a row and column from B to obtain $B/\bar{c}$, it follows that we can view $B/\bar{c}$ as a Ferrers board contained in $[n-1] \times [n-1]$. Then we have that

$$\begin{aligned}
\sum_{k=0}^{n} H_{k,n}(B)x^k &= \sum_{k=0}^{n} r_k(B)(n-k)!(x-1)^k \\
&= \sum_{k=0}^{n} (r_k(B/c) + r_{k-1}(B/\bar{c}))(n-k)!(x-1)^k \\
&= \sum_{k=0}^{n} r_k(B/c)(n-k)!(x-1)^k + (x-1)\sum_{k=1}^{n} r_{k-1}(B/\bar{c})(n-1-(k-1))!(x-1)^{k-1} \\
&= \sum_{k=0}^{n} r_k(B/c)(n-k)!(x-1)^k + (x-1)\sum_{k=0}^{n-1} r_k(B/\bar{c})(n-1-k)!(x-1)^k \\
&= \sum_{k=0}^{n} H_{k,n}(B/c)x^k + \sum_{k=0}^{n-1} H_{k,n-1}(B/\bar{c})x^k.
\end{aligned} \tag{85}$$

Taking the coefficient of x^k on both sides of (85), we obtain the following theorem.

Theorem 21. *Let c be an outside corner square of a Ferrers board $B = F(b_1, \dots, b_n) \subseteq [n] \times [n]$. Then for all $k \geq 0$,*

$$H_{k,n}(B) = H_{k,n}(B/c) + H_{k-1,n-1}(B/\bar{c}) - H_{k,n-1}(B/\bar{c}). \tag{86}$$

We can also give a combinatorial proof of Theorem 21. That is, fix $0 \leq k \leq n$. We partition $E_k = \{\sigma \in S_n : |\sigma \cap B| = k\}$ into two sets: $E_{k,1}$ which is equal to the set of all $\sigma \in E_k$ such that there is a rook in c and $E_{k,0}$ which is equal to the set of all $\sigma \in E_k$ such that there is a no rook in c. Now if start with $\sigma \in E_{k,1}$ and we remove the row and column containing c, then it is easy to see that we will be reduced to a $\tau \in S_{n-1}$ such that $|\tau \cap B/\bar{c}| = k-1$. Thus $|E_{k,1}| = H_{k-1}(B/\bar{c})$ and $H_{k,n}(B) - H_{k-1}(B/\bar{c}) = |E_{k,0}|$.

Similarly, we can partition $F_k = \{\sigma \in S_n : |\sigma \cap B/c| = k\}$ into two sets: $F_{k,1}$ which is equal to the set of all $\sigma \in F_k$ such that there is a rook in c and $F_{k,0}$ which is equal to the set of all $\sigma \in F_k$ such that there is a no rook in c. Again if we start with a $\sigma \in F_{k,1}$ and remove the row and column containing c, then it is easy to see that we will be reduced to a $\tau \in S_{n-1}$ such that $|\tau \cap B/\bar{c}| = k$. Thus $|F_{k,1}| = H_k(B/\bar{c})$ and $H_{k,n}(B/c) - H_k(B/\bar{c}) = |F_{k,0}|$. But it is easy to see that $E_{k,0} = F_{k,0}$ so that we must have

$$H_{k,n}(B) - H_{k-1}(B/\bar{c}) = H_{k,n}(B/c) - H_k(B/\bar{c})$$

which is what we wanted to prove.

Next suppose that $A = F(a_1, \ldots, a_n) \subseteq [n] \times [n]$ is a skyline board and $c = (a_i, i)$ is the top cell of a non-zero column. Then clearly $A/\bar{\bar{c}}$ is another skyline board in $[n] \times [n]$. Then we have that

$$\begin{aligned}
\sum_{k=0}^{n} F_{k,n}(B)x^k &= \sum_{k=0}^{n} f_k(B) n^{n-k}(x-1)^k \\
&= \sum_{k=0}^{n} (f_k(B/c) + f_{k-1}(B/\bar{\bar{c}})) n^{n-k}(x-1)^k \\
&= \sum_{k=0}^{n} f_k(B/c) n^{n-k}(x-1)^k + \frac{(x-1)}{n} \sum_{k=1}^{n} f_{k-1}(B/\bar{\bar{c}}) n^{n-(k-1)}(x-1)^{k-1} \\
&= \sum_{k=0}^{n} f_k(B/c)(n-k)!(x-1)^k + \frac{(x-1)}{n} \sum_{k=0}^{n-1} f_k(B/\bar{\bar{c}}) n^{n-k}(x-1)^k \\
&= \sum_{k=0}^{n} F_{k,n}(B/c)x^k + \sum_{k=0}^{n} F_{k,n}(B/\bar{\bar{c}})x^k. \qquad (87)
\end{aligned}$$

Taking the coefficient of x^k on both sides of (87), we obtain the following theorem.

Theorem 22. *Let c be cell which is at the top of some column of skyline board $A = F(a_1, \ldots, a_n) \subseteq [n] \times [n]$. Then for all $k \geq 0$,*

$$Fk,n(A) = F_{k,n}(A/c) + \frac{1}{n} F_{k-1,n-1}(A/\bar{\bar{c}}) - \frac{1}{n} F_{k,n-1}(A/\bar{\bar{c}}). \qquad (88)$$

We can also give a combinatorial proof of Theorem 22. That is, fix $0 \leq k \leq n$. We partition $U_k = \{f \in \mathbb{F}_n : |f \cap B| = k\}$ into two sets: $U_{k,1}$ which is equal to the set of all $f \in U_k$ such that there is a rook in c and $U_{k,0}$ which is equal to the set of all $f \in E_k$ such that there is a no rook in c. Now suppose c is in column i. If we start with an $f \in E_{k,1}$ and remove the rook and all the cells of A in column i, then we will create a file placement P_f such that P_f contains rooks in each of the columns except i. Clearly $|P \cap (A/\bar{\bar{c}})| = k-1$. Since the i-th column of $A/\bar{\bar{c}}$ is empty, there are n ways to extend P_f by adding a rook in column i to obtain a file placement $g \in \mathbb{F}_n$ such that $|g \cap (A/\bar{\bar{c}})| = k-1$. Thus each $f \in U_{k,1}$ gives rise to n functions counted by $H_{k-1,n}(A/\bar{\bar{c}})$. It follows that $U_{1,k} = \frac{1}{n} H_{k-1,n}(A/\bar{\bar{c}})$ and $F_{k,n}(B) - \frac{1}{n} F_{k-1}(B/\bar{c}) = |U_{k,0}|$.

Similarly, we can partition $V_k = \{f \in \mathbb{F}_n : |f \cap A/c| = k\}$ into two sets: $V_{k,1}$ which is equal to the set of all $f \in V_k$ such that there is a rook in c and $V_{k,0}$ which is equal to the set of all

$f \in V_k$ such that there is a no rook in c. Again each $f \in F_{k,1}$ corresponds to n elements $g \in \mathbb{F}_n$ such that $|g \cap (A/\bar{\bar{c}})| = k$. Thus $|V_{k,1}| = \frac{1}{n}F_k(A/\bar{\bar{c}})$ and $F_{k,n}(B/c) - \frac{1}{n}F_k(A/\bar{\bar{c}}) = |V_{k,0}|$. But it is easy to see that $U_{k,0} = V_{k,0}$ so that we must have

$$F_{k,n}(B) - \frac{1}{n}F_{k-1}(B/\bar{c}) = F_{k,n}(B/c) - \frac{1}{n}F_k(A/\bar{\bar{c}})$$

which is what we wanted to prove.

1.8 A Formula of Frobenius

In this section, we present another way in which the Eulerian numbers arise in rook theory by proving a formula that relates the Stirling numbers of the second kind to the generating function of descents of permutations in S_n which is due to Frobenious, see [18].

To prove this formula, we will consider a new type of board. Let $B = F(b_1, \ldots, b_n)$ be a Ferrers board. We then let the board B_∞ consist of the board B together with infinitely many rows of size n attached below B. We again call the line separating B from the extra rows in B, the bar. We also label the extra row starting from the top with $1, 2, \ldots$. For example, Figure 30 picture B_∞ where $B = F(2, 2, 3, 4, 4)$.

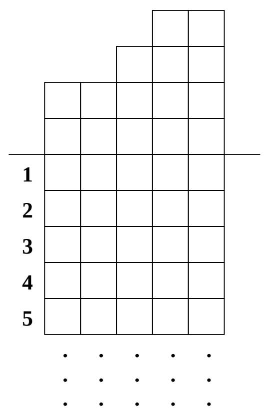

Figure 30: The board B_∞ where $B = F(2, 2, 3, 4, 4)$.

Next let $\mathcal{N}_n(B_\infty)$ denote the set of all placements of n non-attacking rooks in B_∞. For any $P \in \mathcal{N}_n(B_\infty)$, we let $max(P) = k$ if the lowest rook in P is in row k below the bar and we let $max(P) = 0$ if P has no rooks below the bar. We then have the following theorem.

Theorem 23. *Let $B = F(b_1, \ldots, b_n)$ be a Ferrers board. Then*

$$\frac{1}{1-x} \sum_{P \in \mathcal{N}_n(B_\infty)} x^{\max(P)} = \sum_{k \geq 0} x^k \prod_{i=1}^{n} (k + b_i - (i-1)) \tag{89}$$

$$= \sum_{k=0}^{n} \frac{r_{n-k}(B) k! x^k}{(1-x)^{k+1}}. \tag{90}$$

Proof. We shall consider two different ways to sum the LHS of (89). First, it is easy to see that

$$\frac{1}{1-x} \sum_{P \in \mathcal{N}_n(B_\infty)} x^{\max(P)} = \sum_{k \geq 0} x^k |\{P \in \mathcal{N}_n(B_\infty) : max(P) \leq k\}|. \tag{91}$$

However $|\{P \in \mathcal{N}_n(B_\infty) : max(P) \leq k\}|$ is just the number of placements of n non-attacking rooks in B_x where $x = k$. By Theorem 3,

$$|\{P \in \mathcal{N}_n(B_\infty) : max(P) \leq k\}| = \prod_{i=1}^{n}(k + b_i - (i-1)).$$

Thus

$$\frac{1}{1-x} \sum_{P \in \mathcal{N}_n(B_\infty)} x^{\max(P)} = \sum_{k \geq 0} x^k \prod_{i=1}^{n}(k + b_i - (i-1)). \tag{92}$$

On the other hand, we can classify the placements $P \in \mathcal{N}_n(B_\infty)$ by how many rook in P lie above the bar. That is, suppose that we fix a placement Q of $n-k$ rooks in B where $1 \leq k \leq n$. We then wish to compute

$$S_Q(x) = \frac{1}{1-x} \sum_{P \in \mathcal{N}_n(B_\infty); P \cap B = Q} x^{\max(P)}.$$

We can code such a placement $P \in \mathcal{N}_n(B_\infty)$ such that $P \cap B = Q$ by a sequence sequence $(p_1, \ldots, p_k) \in \mathbb{N}^k$ and permutation $\sigma \in S_k$ where p_1 is the distance from row 1 to highest rook in P below the bar, p_2 is the distance between the highest rook in P, etc. and σ is the permutation that corresponds to the rook placement of the the rows and columns that contain rooks below the bar. For example, the rook configuration pictured in Figure 31 would by coded the sequence $(2, 0, 4, 1)$ and the permutation $\sigma = 4\ 1\ 2\ 3$. It is easy to see that if P is coded by $\langle (p_1, \ldots, p_k), \sigma \rangle$, then

$$max(P) = p_1 + \cdots + p_k + k.$$

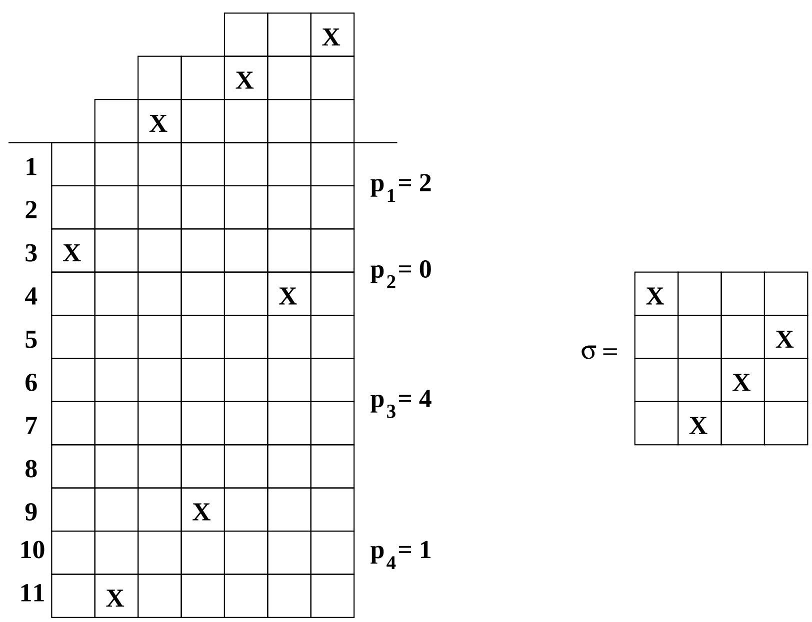

Figure 31: Coding for a rook placement in $\mathcal{N}_n(B_\infty)$.

It follows that

$$\begin{aligned} S_Q(x) &= \frac{1}{1-x} \sum_{P \in \mathcal{N}_n(B_\infty); P \cap B = Q} x^{\max(P)} \\ &= \frac{1}{1-x} \sum_{\sigma \in S_n} \sum_{p_1 \geq 0} \cdots \sum_{p_k \geq 0} x^{p_1 + \cdots + p_k + k} \\ &= \frac{1}{1-x} k! x^k \prod_{i=1}^{k} \sum_{p_i \geq 0} x^{p_i} \\ &= \frac{k! x^k}{(1-x)^{k+1}}. \end{aligned}$$

Since there are clearly $r_n(B)$ rook placements in $\mathcal{N}_n(B_\infty)$ that have no rooks below the bar, we have that

$$\begin{aligned} \frac{1}{1-x} \sum_{P \in \mathcal{N}_n(B_\infty)} x^{\max(P)} &= \frac{1}{1-x} r_n(B) + \sum_{k=1}^{n} \sum_{Q \in \mathcal{N}_{n-k}(B)} S_Q(x) \\ &= \sum_{k=0}^{n} \frac{r_{n-k}(B) k! x^k}{(1-x)^{k+1}} \end{aligned}$$

□

Note that Theorem 23 does not require that $b_n \leq n$. If we start with $B = F(b_1, \ldots, b_n)$ where $b_n \leq n$, then we have that

$$\sum_{k=0}^{n} H_{k,n}(B) x^k = \sum_{k=0}^{n} r_{n-k}(B) k! (x-1)^{n-k}. \tag{93}$$

replacing x by $1/x$ in (93) and then multiplying by x^n gives

$$\sum_{k=0}^{n} H_{k,n}(B) x^{n-k} = \sum_{k=0}^{n} r_{n-k}(B) k! x^k (1-x)^{n-k}. \tag{94}$$

Then dividing (94) by $(1-x)^{n+1}$ yeilds

$$\frac{\sum_{k=0}^{n} H_{k,n}(B) x^{n-k}}{(1-x)^{n+1}} = \sum_{k=0}^{n} \frac{r_{n-k}(B) k! x^k}{(1-x)^{k+1}}. \tag{95}$$

Thus we have the following corollary.

Corollary 2. *Let $B = F(b_1, \ldots, b_n)$ be a Ferrers board where $b_n \leq n$.*

$$\begin{aligned} \frac{1}{1-x} \sum_{P \in \mathcal{N}_n(B_\infty)} x^{\max(P)} &= \sum_{k=0}^{n-1} \frac{r_{n-k}(B) k! x^k}{(1-x)^{k+1}} \\ &= \frac{\sum_{k=0}^{n} H_{k,n}(B) x^{n-k}}{(1-x)^{n+1}} \end{aligned}$$

Now consider the special case where $B = St_n$, then we know that $r_{n-k}(St_n) = S_{n,k}$ and $H_{n,k}(St_n)$ is the number of permutations of S_n with k descents. For any permutation, $\sigma = \sigma_1 \cdots \sigma_n \in S_n$, we let the complement of σ, σ^c, be defined by

$$\sigma^c = (n+1-\sigma_1)\cdots(n+1-\sigma_n).$$

It is then easy to see that σ has k descents if and only if σ^c has $n-1-k$ descents so that $H_{n,k}(St_n)$ is the number of permutations of S_n with $n-1-k$ descents. It then follows that

$$\sum_{k=0}^{n-1} H_{k,n}(St_n)x^{n-k} = \sum_{\sigma\in S_n} x^{\mathrm{des}(\sigma)+1}. \tag{96}$$

Combining (96) with Corollary 2 we have the following formula of Frobenius.

Theorem 24.

$$\frac{\sum_{\sigma\in S_n} x^{\mathrm{des}(\sigma)+1}}{(1-x)^{n+1}} = \sum_{k=1}^{n} \frac{S_{n,k}k!x^k}{(1-x)^{k+1}}. \tag{97}$$

Next consider the special case where $B = F(b_1,\ldots,b_n)$ where $b_i = n-1$ for all i. Thus $B = \mathcal{L}_n$ is the Laguerre board and $r_{n-k}(\mathcal{L}_n) = L_{n,k}$. Clearly $|\sigma \cap \mathcal{L}_n| = n-1$ for all $\sigma \in S_n$, so that

$$\sum_{k=0}^{n-1} H_{n,k}(\mathcal{L}_n)x^{n-1} = n!x. \tag{98}$$

Hence corollary 2 then yields the following theorem

Theorem 25.

$$\frac{n!x}{(1-x)^{n+1}} = \sum_{k=1}^{n} \frac{L_{n,k}k!x^k}{(1-x)^{k+1}}. \tag{99}$$

Next we prove an analogue of Theorem 23 for file placements. Given a skyline board $B = F(b_1, ldots, b_n)$, let $\mathcal{C}_n(B_\infty)$ denote the set of all file placements of n rooks in B_∞. For any $P \in \mathcal{C}_n(B_\infty)$, we let $max(P) = k$ if the lowest rook in P is in row k below the bar and we let $max(P) = 0$ if P has no rooks below the bar. We then have the following theorem.

Theorem 26. *Let $B = F(b_1,\ldots,b_n)$ be a skyline board. Then*

$$\frac{1}{1-x}\sum_{P\in\mathcal{C}_n(B_\infty)} x^{\max(P)} = \sum_{k\geq 0} x^k \prod_{i=1}^{n}(k+b_i) \tag{100}$$

$$= \frac{f_n(B)}{1-x} + \sum_{k=1}^{n} f_{n-k}(B)\sum_{j=1}^{k}\frac{j!S_{n,j}}{(1-x)^{j+1}}. \tag{101}$$

Proof. It is easy to see that the sum $\frac{1}{1-x}\sum_{P\in\mathcal{C}_n(B_\infty)} x^{\max(P)}$ is invariant if we permute the columns so that there is no loss in generality in assuming the B is a Ferrers board. We shall consider two different ways to sum the LHS of (89). As in Theorem 23, it is easy to see that

$$\frac{1}{1-x}\sum_{P\in\mathcal{C}_n(B_\infty)} x^{\max(P)} = \sum_{k\geq 0} x^k |\{C \in \mathcal{C}_n(B_\infty) : max(P) \leq k\}|. \tag{102}$$

However $|\{P \in \overline{\mathcal{C}}_n(B_\infty) : max(P) \leq k\}|$ is just the number of file placements of n rooks in B_x where $x = k$. By Theorem 4,

$$|\{P \in \mathcal{C}_n(B_\infty) : max(P) \leq k\}| = \prod_{i=1}^{n}(k + b_i).$$

Thus

$$\frac{1}{1-x} \sum_{P \in \mathcal{C}_n(B_\infty)} x^{\max(P)} = \sum_{k \geq 0} x^k \prod_{i=1}^{n}(k + b_i). \tag{103}$$

On the other hand, we can classify the file placements $P \in \mathcal{C}_n(B_\infty)$ by how many rook in P lie above the bar. That is, suppose that we fix a placement Q of $n-k$ rooks in B where $1 \leq k \leq n$. We then wish to compute

$$T_Q(x) = \frac{1}{1-x} \sum_{P \in \mathcal{C}_n(B_\infty); P \cap B = Q} x^{\max(P)}.$$

Now if $P \in \mathcal{C}_n(B_\infty)$ and $P \cap B = Q$, then there must be k rooks in P which lie below the bar. However because we are considering file placements, the number of rows which can contain rooks below the bar can be any j where $1 \leq j \leq k$. So fix j with $1 \leq j \leq k$ and consider all the set of all file placements $P \in \mathcal{C}_n(B_\infty)$ such that $P \cap B = Q$ and there are j rooks below the bar which contain rooks in P. We can code such a placements P by a sequence sequence $(p_1, \ldots, p_j) \in \mathbb{N}^j$ and a surjective function f from $[k]$ onto j. That is, we p_1 be the distance from row 1 to highest row that contain rooks in P below the bar, p_2 be the distance between the highest row containing rooks in P below the bar and the second highest row that contains rooks in P below the bar, etc. in P, etc. The f is the function that corresponds to the file placement of the the rows and columns that contain rooks below the bar. For example, the file placement pictured in Figure 32 would by coded the sequence $(2, 5)$ and the permutation f where $f(1) = f(4) = 2$, $f(2) = f(3) = 1$. If $f : [k] \rightarrow [j]$ is a surjective function, then clearly we can code f by the ordered set partition of $[k]$ consisting of $(f^{-1}(1), \ldots, f^{-1}(j))$ so that there are $j!S_{k,j}$ such functions. It is easy to see that if P is coded by $\langle (p_1, \ldots, p_j), f \rangle$, then

$$max(P) = p_1 + \cdots + p_j + j.$$

It follows that

$$\begin{aligned}
T_Q(x) &= \frac{1}{1-x} \sum_{P \in \mathcal{C}_n(B_\infty); P \cap B = Q} x^{\max(P)} \\
&= \frac{1}{1-x} \sum_{j=1}^{k} \sum_{f:[k]\rightarrow[j] \text{ is surjective}} \sum_{p_1 \geq 0} \cdots \sum_{p_j \geq 0} x^{p_1 + \cdots + p_k + k} \\
&= \frac{1}{1-x} \sum_{j=1}^{k} j!S_{k,j} x^j \prod_{i=1}^{j} \sum_{p_i \geq 0} x^{p_i} \\
&= \sum_{j=1}^{k} \frac{j!S_{k,j} x^j}{(1-x)^{j+1}}.
\end{aligned}$$

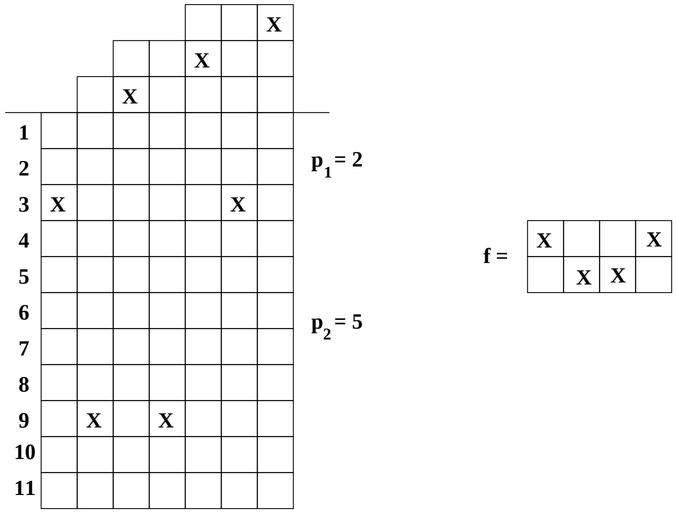

Figure 32: Coding for a file placement in $\mathcal{C}_n(B_\infty)$.

Clearly there are $f_n(B)$ file placements in $\mathcal{C}_n(B_\infty)$ that have no rooks below the bar. Thus

$$\begin{aligned}
\frac{1}{1-x} \sum_{P \in \mathcal{C}_n(B_\infty)} x^{\max(P)} &= \frac{f_n(B)}{1-x} + \sum_{k=1}^{n} \sum_{Q \in \mathcal{N}_{n-k}(B)} T_Q(x) \\
&= \frac{f_n(B)}{1-x} + \sum_{k=1}^{n} f_{n-k}(B) \sum_{j=1}^{k} \frac{j! S_{k,j} x^j}{(1-x)^{j+1}}.
\end{aligned}$$

□

It is natural to ask whether there is an analogue of Corollary 2 for file placements when $B = F(b_1, \ldots, b_n)$ is a Ferrers board contained $[n] \times [n]$. That is, it follows from Theorem 26 that if $f_n(B) = 0$, then

$$\frac{1}{1-x} \sum_{P \in \mathcal{C}_n(B_\infty)} x^{\max(P)} = \frac{\sum_{k=1}^{n} f_{n-k}(B) \sum_{j=1}^{k} j! S_{k,j} x^j (1-x)^{n-j}}{(1-x)^{n+1}}. \tag{104}$$

Hence one can ask whether the polynomial

$$U_{B,n}(x) = \sum_{k=1}^{n} f_{n-k}(B) \sum_{j=1}^{k} j! S_{k,j} x^j (1-x)^{n-j} \tag{105}$$

is a polynomial with non-negative integer coefficients. This is not true in general. For example, if $B = F(0, 2)$, then one can easily calculate that $U_{B,n}(x) = 3x - x^2$. However, we can identify a large set of boards where $F_{B,n}(x) \in \mathbb{N}[x]$. That is, suppose that $B = F(b_1, \ldots, b_n)$ is a Ferrers board where $b_1 = 0$ and $b_{i+1} \leq b_i + 1$. Then consider the sequence $(a_1, \ldots, a_n) = (b_n, \ldots, b_1) + (0, 1, 2, \ldots, n-1)$, i.e. $a_i = b_{n+1-i} + i - 1$ for $i = 1, \ldots, n$. Since our assumptions on the sequence $(b_1, \ldots, b_n)$ force that $b_{n+1-i} \leq n - i$, we must have $a_i \leq n - 1$ for all i. Moreover, it is easy to check that our assumption that $b_{i+1} \leq b_i + 1$ ensures that $a_i \leq a_{i+1}$ for all $i = 1, \ldots, n-1$. Thus $A = F(a_1, \ldots, a_n)$ is a Ferrers board contained in $[n-1] \times [n]$ and, hence, $r_n(A) = 0$. Moreover, it is easy to see that $\prod_{i=1}^n (k + a_i - (i-1)) = \prod_{i=1}^n (k + b_i)$. But then if follows from Theorems 23 and 26 and Corollary 2 that

$$\begin{aligned}
\frac{1}{1-x} \sum_{P \in \mathcal{N}_n(A_\infty)} x^{\max(P)} &= \frac{\sum_{k=0}^n H_{k,n}(A) x^{n-k}}{(1-x)^{n+1}} \\
&= \sum_{k \geq 1} x^k \prod_{i=1}^n (k + a_i - (i-1)) \\
&= \sum_{k \geq 1} x^k \prod_{i=1}^n (k + b_i) \\
&= \frac{1}{1-x} \sum_{P \in \mathcal{C}_n(B_\infty)} x^{\max(P)} \\
&= \frac{\sum_{k=1}^n f_{n-k}(B) \sum_{j=1}^k j! S_{k,j} x^j (1-x)^{n-j}}{(1-x)^{n+1}}.
\end{aligned}$$

Thus $U_{B,n}(x) = \sum_{k=0}^n H_{k,n}(A) x^{n-k}$ in this case. Thus we have proved the following.

Theorem 27. *Suppose that $B = F(b_1, \ldots, b_n)$ is a Ferrers board such that $b_1 = 0$ and $b_{i+1} \leq b_i + 1$ for all i. Then if $(a_1, \ldots, a_n) = (b_n, \ldots, b_1) + (0, 1, 2, \ldots, n-1)$, $F(a_1, \ldots, a_n)$ is a Ferrers board contained in $[n-1] \times [n]$ and*

$$\sum_{k=0}^n H_{k,n}(A) x^{n-k} = \sum_{k=1}^n f_{n-k}(B) \sum_{j=1}^k j! S_{k,j} x^j (1-x)^{n-j}. \tag{106}$$

Consider the special case of Theorem 106 where $B = St_n = F(0, 1, 2, \ldots, n-1)$. In that case, $A = (n-1, \ldots, n-1) = \mathcal{L}_n$. By (98), we have $\sum_{k=0}^{n-1} H_{n,k}(\mathcal{L}_n) x^{n-1} = n! x$. Thus in this case, we have the following analogue of the Frobenius formula for file numbers.

Corollary 3. *Let $St_n = F(0, 1, 2, \ldots, n-1)$. Then*

$$\frac{1}{1-x} \sum_{P \in \mathcal{C}_n(B_\infty)} x^{\max(P)} = \frac{n! x}{(1-x)^{n+1}}. \tag{107}$$

1.9 Algebraic Identities for Ferrers Boards

If $B = B(c_1, \ldots, c_n)$ is a Ferrers board, let

$$PR(x, B) = \prod_{i=1}^n (x + c_i - i + 1). \tag{108}$$

Recall that by Theorem 3, we have that

$$\sum_{k=0}^n x(x-1) \cdots (x-k+1) r_{n-k}(B) = PR(x, B). \tag{109}$$

We can use this formula to prove the following proposition

Proposition 2. *Let $B = B(c_1, \ldots, c_n)$ be a Ferrers board. Then*

$$k!r_{n-k}(B) = \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} PR(j, B) \tag{110}$$

$$H_{n-k,n}(B) = \sum_{j=0}^{k} \binom{n+1}{k-j} (-1)^{k-j} PR(j, B). \tag{111}$$

Proof. By (109), the RHS of (110) equals

$$\sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \sum_{s} \binom{j}{s} s!r_{n-s} \tag{112}$$

$$= \sum_{s\geq 0} s!r_{n-s} \sum_{j\geq s} \binom{k}{j} (-1)^{k-j} \binom{j}{s}$$

$$= \sum_{s\geq 0} s!r_{n-s} \frac{(1-z)^k}{(1-z)^{s+1}}|_{z^{k-s}} \tag{113}$$

$$= \sum_{s\geq 0} s!r_{n-s}\delta_{s,k},$$

where

$$\delta_{k,j} = \begin{cases} 1 \text{ if } k = j \\ 0 \text{ else.} \end{cases}$$

Also using (109), the RHS of (111) equals

$$\sum_{j=0}^{k} \binom{n+1}{k-j} (-1)^{k-j} \sum_{s\leq j} \binom{j}{s} s!r_{n-s} \tag{114}$$

$$= \sum_{s} s!r_{n-s} \sum_{j\geq s} \binom{n+1}{k-j} (-1)^{k-j} \binom{j}{s}$$

$$= \sum_{s} s!r_{n-s} \frac{(1-x)^{n+1}}{(1-x)^{s+1}}|_{x^{k-s}}$$

by the binomial theorem. Now use (2). ☐

A unitary vector is a nonzero vector all of whose coordinates are 0 or 1. For a vector v of nonnegative integers, let $g_k(v)$ denote the number of ways of writing v as a sum of k unitary vectors. By convention we set $g_0(0) = 0$. For example, $g_2(2, 1) = 2$ and $g_3(2, 1) = 3$ since

$$\begin{aligned}(2, 1) &= (1, 1) + (1, 0) \\ &= (1, 0) + (1, 1) \\ &= (1, 0) + (1, 0) + (0, 1) \\ &= (1, 0) + (0, 1) + (1, 0) \\ &= (0, 1) + (1, 0) + (1, 0).\end{aligned} \tag{115}$$

Letting 1^n stand for the vector with n ones, it is easy to see that $g_k(1^n) = k!S(n,k)$, where $S(n,k)$ is the Stirling number of the second kind. MacMahon derived a number of identities for $g_k(v)$ in connection with his work on Simon Newcomb's problem. Here we show how these numbers can be connected with rook theory. We let $n = v_1 + \dots v_p$.

Theorem 28. *For any* v,

$$g_k(v) = \frac{k! r_{n-k}(G_v)}{\prod_i v_i!}. \tag{116}$$

Proof. By definition we have

$$\sum_v \prod_i x_i^{v_i} g_k(v) = (\prod_i (1+x_i) - 1)^k. \tag{117}$$

Hence

$$\begin{aligned} \prod_i (1+x_i)^z &= \sum_{k \geq 0} \binom{z}{k} (\prod_i (1+x_i) - 1)^k \qquad (118) \\ &= \sum_{k \geq 0} \binom{z}{k} \sum_w \prod_i x_i^{w_i} g_k(w). \end{aligned}$$

Taking the coefficient of $\prod_i x_i^{v_i}$ on both sides above yields

$$\prod_i \binom{z}{v_i} = \sum_{k \geq 0} \binom{z}{k} g_k(v). \tag{119}$$

Next note that $PR(z, G_v) = \prod_i v_i! \binom{z}{v_i}$. Comparing (119) with the $B = G_v$ case of (109) we obtain (116). $\square$

Corollary 4.

$$\sum_k g_k(v) x^{n-k} = \sum_j N_{j+1}(v)(x+1)^j. \tag{120}$$

Proof. This follows from (116), (79), and (2). We also provide a direct combinatorial proof, which is based on a argument in [**?**, p. 61] proving a closely related identity. By comparing coefficients of x^{n-k} on both sides of (120) we get

$$g_k(v) = \sum_j N_{j+1}(v) \binom{j}{n-k}. \tag{121}$$

To prove (121), start with a unitary composition C into k parts, say

$$w_1 + w_2 + \dots + w_k = v. \tag{122}$$

For each vector w_i in C, associate a subset $S(w_i)$ by letting $p \in S(w_i)$ iff $w_{i,p} = 1$. For example, if $z = (1,0,0,1,1,0,1)$, $S(z) = \{1,4,5,7\}$. Next form a multiset permutation $M(C)$ with bars between some elements by listing the elements of $S(w_1)$ in decreasing order, followed by a bar and then the elements of $S(w_2)$, in decreasing order, followed by a bar, ..., followed by the elements of $S(w_k)$, in decreasing order. If C is the composition

$$\begin{aligned}(1,1,0,0,0,0)+(1,0,0,0,0,0)+(0,0,1,0,0,0)+(0,0,0,1,0,0)+(0,0,0,0,0,1)\\+(1,1,0,0,1,0)+(0,0,0,1,0,0)+(0,0,0,1,0,0)+(0,1,0,0,0,0)\end{aligned}$$

then $M(C)=21|1|3|4|6|521|4|4|2$. Note that if $M(C)$ has j descents, then we have bars at each of the $n-1-j$ non-descents, together with an additional $j-n+k$ bars at descents for a total of $n-1-j+j-n+k=k-1$ bars. Thus we have a map from unitary compositions with k parts to multiset permutations with say j descents, with an additional $j-n+k$ bars chosen from the descents, which is counted by the RHS of (121). It is easy to see the map is invertible. □

Theorem 29. *For any Ferrers board B,*

$$\sum_{j=0}^{\infty} PR(j,B)z^j = \frac{\sum_{k=0}^{n} z^k H_{n-k,n}(B)}{(1-z)^{n+1}}. \tag{123}$$

Proof.

$$\begin{aligned}(1-z)^{n+1}\sum_{j=0}^{\infty} PR(j,B)z^j|_{z^k} &= \sum_{j=0}^{k}\binom{n+1}{k-j}(-1)^{k-j}PR(j,B) \\ &= H_{n-k,n}(B)\end{aligned} \tag{124}$$

by (111). □

Letting $v=1^n$ in (123) we get

Corollary 5.

$$\frac{\sum_{k=0}^{n} z^k N_{k+1}(1^n)}{(1-z)^{n+1}} = \sum_{j=0}^{\infty} z^j j^n. \tag{125}$$

Theorem 30. *For any Ferrers board B,*

$$\sum_{k=0}^{n}\binom{x+k}{n}H_{k,n}(B) = PR(x,B). \tag{126}$$

Proof. It suffices to prove (126) under the assumption that $x\in\mathbb{N}$. Then the RHS of (126) equals

$$\begin{aligned}\left(\sum_{k=0}^{\infty} y^k PR(k,B)\right)|_{y^x} &= \left(\frac{\sum_j H_{n-j,n}(B)y^j}{(1-y)^{n+1}}\right)|_{y^x} \\ &= \left(\sum_j H_{n-j,n}(B)y^j\right)\left(\sum_{m=0}^{\infty} y^m\binom{n+m}{m}\right)|_{y^x} \\ &= \sum_{j=0}^{x} H_{n-j,n}(B)\binom{n+x-j}{x-j} \\ &= \sum_{k\geq 0} H_{k,n}(B)\binom{x+k}{n}.\end{aligned} \tag{127}$$

□

By letting $B = G_v$ in (126) and using (79) we get

Corollary 6. *For any $v \in \mathbb{N}^p$,*

$$\sum_{k=0}^{n} \binom{x+k}{n} N_{k+1}(v) = \prod_{i=1}^{p} \binom{x}{v_i}. \tag{128}$$

Remark 1. *When $v = 1^n$, (128) is known as Worpitsky's identity.*

1.10 Vector Compositions

For $v \in \mathbb{N}^p$, let $f_k(v)$ denote the number of ways of writing

$$v = w_1 + \ldots + w_k, \tag{129}$$

where $w_i \in \mathbb{N}^p$ with $|w_i| = \sum_j w_{ij} > 0$. For example if $v = (2,1)$, in addition to the ways of decomposing v into unitary vectors as in (115), we have

$$(2,1) = (2,1) \tag{130}$$
$$= (2,0) + (0,1) = (0,1) + (2,0), \tag{131}$$

so $f_1(2,1) = 1$, $f_2(2,1) = 4$, and $f_3(2,1) = 3$. MacMahon first defined and studied $f_k(v)$, deriving of (132) and (136) below.

Proposition 3. *For any $v \in \mathbb{N}^p$,*

$$\prod_i \binom{z + v_i - 1}{v_i} = \sum_{k \geq 0} \binom{z}{k} f_k(v), \tag{132}$$

where we define $f_0(v) = \delta_{n,0}$.

Proof. By definition we have

$$\sum_v \prod_i x_i^{v_i} f_k(v) = (\prod_i \frac{1}{(1-x_i)} - 1)^k. \tag{133}$$

Hence

$$\begin{aligned}(\prod_i \frac{1}{(1-x_i)})^z &= \sum_{k \geq 0} \binom{z}{k} (\prod_i \frac{1}{(1-x_i)} - 1)^k \\ &= \sum_{k \geq 0} \binom{z}{k} \sum_w \prod_i x_i^{w_i} f_k(w).\end{aligned} \tag{134}$$

Taking the coefficient of $\prod_i x_i^{v_i}$ on both sides above yields (132). $\square$

Corollary 7. *Let F_v be the Ferrers board whose first v_1 columns are of height $v_1 - 1$, whose next v_2 columns are of height $v_1 + v_2 - 1$, ..., and whose last v_p columns are of height $v_1 + \ldots + v_p - 1$, so $PR(z, F_v) = \prod_i v_i! \binom{z+v_i-1}{v_i}$. Then*

$$f_k(v) = \frac{k! r_{n-k}(F_v)}{\prod_i v_i!}. \tag{135}$$

Theorem 31. *(MacMahon [42,]).*

$$\sum_k f_k(v)x^{n-k} = \sum_j N_j(v)(x+1)^{n-j}. \tag{136}$$

Exercise 1. *Prove (136) combinatorially using an argument similar to the one above proving (120).*

By combining (2), (135) and (136) we obtain

Corollary 8.

$$N_k(v) = \frac{1}{\prod_i v_i!} H_{n-k,n}(F_v). \tag{137}$$

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