Pfaffians of Toeplitz payoff matrices

Ron Evans
Department of Mathematics
University of California at San Diego
La Jolla, CA 92093-0112
revans@ucsd.edu

Nolan Wallach
Department of Mathematics
University of California at San Diego
La Jolla, CA 92093-0112
nwallach@ucsd.edu

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Abstract

The purpose of this paper is to evaluate the Pfaffians of certain Toeplitz payoff matrices associated with integer choice matrix games.

1
1 Introduction

For \( w \in \mathbb{R} \), let \( M(n, k, w) \) denote the \( n \times n \) skew-symmetric Toeplitz matrix whose \( k \) superdiagonals in the upper right corner have all entries \(-w\), and whose remaining \( n - k - 1 \) superdiagonals have all entries 1. For example, \( M(6, 3, w) \) is the matrix

\[
\begin{bmatrix}
0 & 1 & 1 & -w & -w & -w \\
-1 & 0 & 1 & 1 & -w & -w \\
-1 & -1 & 0 & 1 & 1 & -w \\
w & -1 & -1 & 0 & 1 & 1 \\
w & w & -1 & -1 & 0 & 1 \\
w & w & w & -1 & -1 & 0
\end{bmatrix}.
\]

In earlier work [4], the Pfaffians of (scalar multiples of) \( M(2n, 2n - 2, w) \) were evaluated. Our main result (Theorem 2.4) evaluates the Pfaffians of the matrices \( M(2n, m, w) \) for all \( n \geq 1 \) with \( n \geq m \geq 0 \). Up to sign, these Pfaffians turn out to be the polynomials \( F_m(w) \) defined in (2.1). We remark that \( F_m(w) \) is the difference of two Chebyshev polynomials of the second kind, namely

\[
F_m(w) = U_m((w + 1)/2) - U_{m-1}((w + 1)/2).
\]

A connection between Chebyshev polynomials of the second kind and Pfaffians of some skew-symmetric Toeplitz band matrices may be found in [2]. For a paper involving Pfaffians of general skew-symmetric Toeplitz matrices, see [5].

The matrices \( M(n, k, w) \) arise in connection with the following integer choice matrix game. Rowan secretly chooses an integer \( i \in \{1, 2, \ldots, n\} \) and his opponent Calum secretly chooses an integer \( j \in \{1, 2, \ldots, n\} \). If \( i = j \), the game is a draw. If \( i \neq j \), the player with the smallest integer wins one dollar from his opponent, unless \(|i - j| \geq n - k\), in which case the player with the smallest integer pays \( w \) dollars to his opponent. The matrix \( M(n, k, w) \) is the payoff matrix for Rowan, when identifying Rowan’s \( i \) with the \( i \)-th row and Calum’s \( j \) with the \( j \)-th column.

Let \( \mathbf{p} = (p_1, p_2, \ldots, p_n) \) be a vector of probabilities \( p_i \) with \( p_1 + \cdots + p_n = 1 \) and \( p_i \geq 0 \). If \( \mathbf{p} \in \ker M(n, k, w) \), then \( \mathbf{p} \) is an optimal strategy for this payoff matrix. If further \( \ker M(n, k, w) = \mathbb{R} \mathbf{p} \) and each \( p_i > 0 \), then the optimal strategy \( \mathbf{p} \) is unique [4, Prop. 2.1]. To simplify future notation,
we will identify an optimal strategy \( p \) with any positive scalar multiple of \( p \). An example of an optimal strategy for the payoff matrix \( M(6, 3, w) \) with \( w = 2 \cos(\pi/7) - 1 \) is \( p = (w^2, w, 1 - w^2, w, w^2, 0) \). That optimal strategy is not unique, as another is \( (w^2 + w - 1, 1, w, w, 1, w^2 + w - 1) \).

From here on, we will assume that \( n \geq 1 \) and \( n \geq m \geq 0 \). In Lemma 2.1, we prove that the dimension of \( \text{ker } M(2n+1, m, w) \) is 1 for each \( w \in \mathbb{R} \). The special case \( m = n \) was already proved in [3, Lemma 1]. It turns out that \( \text{ker } M(2n+1, m, w) \) is generated by a vector whose entries up to sign are all Pfaffians of matrices of the form \( M(2k, k, w) \) with \( k \leq m \).

In Lemma 2.3, we apply Lemma 2.1 to evaluate the determinant of \( M(2n, m, w) \). This determines the Pfaffian up to sign. Finally, we determine the sign in Theorem 2.4.

2 Pfaffians

As in [3], define monic polynomials \( F_m := F_m(w) \) of degree \( m \) by

\[
F_0 = 1, \quad F_1 = w, \quad F_m = (w + 1)F_{m-1} - F_{m-2}, \quad \text{for } m > 1.
\]

It is convenient to also set \( F_{-1} = 1 \). For example, \( F_{-1} = F_0 = 1, F_1 = w, F_2 = w^2 + w - 1, F_3 = w^3 + 2w^2 - w - 1, F_4 = w^4 + 3w^3 - 3w, \) and \( F_5 = w^5 + 4w^4 + 2w^3 - 5w^2 - 2w + 1 \). As noted in [3], for each \( m \geq 1 \), the \( m \) zeros of \( F_m \) are

\[
2 \cos(h\pi/(2m + 1)) - 1, \quad h = 1, 3, 5, \ldots, 2m - 1.
\]

In particular, the zeros of \( F_m \) are distinct and they lie in the interval \((-3, 1)\).

For each \( m \geq 0 \), define the sequences

\[
S_m = \begin{cases} F_{m-1}, F_{m-3}, F_{m-5}, \ldots, F_0, & \text{for odd } m \\ F_{m-1}, F_{m-3}, F_{m-5}, \ldots, F_1, & \text{for even } m \end{cases}
\]

and

\[
T_m = \begin{cases} F_1, F_3, F_5, \ldots, F_{m-2}, & \text{for odd } m \\ F_0, F_2, F_4, \ldots, F_{m-2}, & \text{for even } m \end{cases}
\]

and let \( S'_m \) and \( T'_m \) denote the respective sequences written in reverse order. Interpret \( S_m \) and \( T_m \) as empty sequences when \( m = 0 \). Note that the
combined sequence $S_m, T_m$ consists of the $m$ distinct terms $F_0, F_1, \ldots, F_{m-1}$. Define the alternating sequence $V_{n,m}$ of length $2n + 1 - 2m$ by

$$V_{n,m} = F_m, -F_m, F_m, -F_m, \ldots, F_m.$$ 

Define the column vector $Y_{n,m}$ of length $2n + 1$ by

$$Y_{n,m} = Y_{n,m}(w) = (S_m, T_m, V_{n,m}, T'_m, S'_m).$$

For example, when $n = m$ (in which case the sequence $V_{n,m}$ consists of the single term $F_m$),

$$Y_{0,0} = (F_0), \quad Y_{1,1} = (F_0, F_1, F_0), \quad Y_{2,2} = (F_1, F_0, F_2, F_0, F_1),$$

$$Y_{3,3} = (F_2, F_0, F_1, F_3, F_1, F_0, F_2),$$

$$Y_{4,4} = (F_3, F_1, F_0, F_2, F_4, F_2, F_0, F_3),$$

$$Y_{5,5} = (F_4, F_2, F_0, F_1, F_3, F_5, F_3, F_1, F_0, F_2, F_4).$$

The following lemma shows that the matrix $M(2n+1, m, w)$ has a nullspace of dimension 1 generated by the vector $Y_{n,m}$.

**Lemma 2.1.** For each $w \in \mathbb{R}$ and all $n \geq 1$ with $n \geq m \geq 0$, we have

$$\ker M(2n+1, m, w) = \mathbb{R}Y_{n,m}(w).$$

**Proof.** Our proof follows closely the proof in [3, Lemma 1]. We first show that $Y_{n,m}$ is in the nullspace of $M(2n+1, m, w)$ by showing that

$$M_iY_{n,m} = 0 \quad \text{for} \quad 1 \leq i \leq 2n+1,$$

where $M_i$ denotes the $i$-th row of $M(2n+1, m, w)$. For $2 \leq i \leq 2n+1$, the vector $M_i - M_{i-1}$ either has exactly three nonzero entries or else has exactly two nonzero entries which are consecutive and equal to $-1$. In the first case, the three term recurrence in (2.1) shows that

$$M_i - M_{i-1})Y_{n,m} = 0.$$

In the second case, the two consecutive $-1$’s occur in the central portion of length $2n + 1 - 2m$ in the vector $M_i - M_{i-1}$. Since the signs in the sequence $V_{n,m}$ are alternating, it follows that (2.4) holds in this second case as well. The vector $M_{n+1}$ begins with $n$ entries $-1$, followed by 0, ending with $n$
entries 1. Thus $M_{n+1}Y_{n,m} = 0$. Together with (2.4), this completes the proof of (2.3).

It remains to prove that $M(2n + 1, m, w)$ has nullity 1. The case $m = 0$ follows immediately from the special case $\nu = -1$ of [3, Lemma 1]. Thus we may stipulate that $m \geq 1$. Assume for the purpose of contradiction that for some real $w$, there exists $m$ taken minimal for which the nullity of $M(2n + 1, m, w)$ exceeds 1. Since skew-symmetric matrices have even rank, the nullity must be at least 3. Thus there is a nonzero vector $C$ in the nullspace of $M(2n + 1, m, w)$ whose middle and last entries are both 0. Let $\bar{C}$ be the vector of length $2n - 1$ obtained from $C$ by deleting the middle and last entries. The matrix $M(2n - 1, m - 1, w)$ is obtained upon deleting the middle and last rows and columns from $M(2n + 1, m, w)$. Therefore $M(2n - 1, m - 1, w)\bar{C} = 0$. Replacing $C$ by a scalar multiple of $C$ if necessary, we may assume that $\bar{C} = Y_{n-1,m-1}$, in view of the minimality of $m$. Thus $C = (\ldots, 0, \ldots, 0)$, where the first blank is filled with the first $n$ entries of $Y_{n-1,m-1}$ and the second blank is filled with the remaining $n - 1$ entries of $Y_{n-1,m-1}$. We have $M_{n+1}C = 0$, and everything cancels in the dot product on the left except for the middle entry $(-1)^{n-m}F_{m-1}$ of $Y_{n-1,m-1}$. Thus $F_{m-1} = 0$. Since also $M_{2n+1}C = 0$, we have

$$(w - 1)(F_0 + F_1 + \cdots + F_{m-1}) + F_{m-1} = 0.$$ The left side above equals $F_m$, as is easily seen by induction. Thus $F_m = 0$, and since $F_{m-1} = 0$ as well, we obtain a contradiction to (2.1).

We now mention an application of Lemma 2.1 to matrix games. First suppose that $n = m$. By (2.2), all the entries of $Y_{n,m}(w)$ are positive when $w > 2 \cos(\pi/(2m + 1) - 1$. For every such $w$, it follows from Lemma 2.1 that $Y_{n,m}(w)$ is the unique optimal strategy for the payoff matrix $M(2n + 1, m, w)$. Now suppose that $n > m$. Then it is no longer true that $Y_{n,m}(w)$ is a strategy for $M(2n + 1, m, w)$ when $w > 2 \cos(\pi/(2m + 1) - 1$, because positivity fails due to the alternating signs in the sequence $V_{n,m}$. On the other hand, when $w = 2 \cos(\pi/(2m + 1) - 1$, then all entries in $Y_{n,m}(w)$ are nonnegative, so that $Y_{n,m}(w)$ is indeed an optimal strategy for $M(2n + 1, m, w)$, although not unique. For example, take $n = 4$, $m = 3$, and $w = 2 \cos(\pi/7) - 1$. Then

$$Y_{4,3}(w) = (w^2 + w - 1, 1, w, 0, 0, 0, w, 1, w^2 + w - 1)$$

is an optimal strategy, and so is

$$(0, w + 1, w^2 + 2w, 0, 0, 0, 1, w^2 + 2w, w + 1).$$
We will need the following simple lemma for the proof of Lemma 2.3.

**Lemma 2.2.** Let $E(k)$ denote the $k \times k$ skew-symmetric matrix with all entries above the diagonal equal to 1. Then the determinant $d(k)$ of $E(k)$ is 1 or 0 according as $k$ is even or odd.

**Proof.** Denote the $j$-th row of $E(k)$ by $E_j$. In the matrix $E(k)$, first replace $E_j$ by $E_j - E_{j+1}$ for each $j = 1, 2, \ldots, k - 1$. Each of the first $k - 1$ rows now has two nonzero entries, which are consecutive and equal to 1. To evaluate $d(k)$, expand along the first column to see that

$$d(k) = d(k - 1) + (-1)^k.$$

The result now follows by induction. \qed

The next lemma shows that up to sign, the Pfaffian of $M(2n, m, w)$ is $F_m(w)$. The sign will be determined in Theorem 2.4.

**Lemma 2.3.** Let $D(w)$ denote the determinant of $M(2n, m, w)$. For each $w \in \mathbb{R}$ and all $n \geq 1$ with $n \geq m \geq 0$, we have $D(w) = F_m(w)^2$.

**Proof.** Let $\tilde{Y}_{n,m}$ be the vector of length $2n$ obtained from $Y_{n,m}$ by deleting the middle entry $(-1)^{n-m}F_m$. One obtains the matrix $M(2n+1, m, w)$ upon deleting the middle row and column from $M(2n+1, m, w)$. Thus by Lemma 2.1,

$$M(2n, m, w)\tilde{Y}_{n,m}(w) = Z_{n,m}(w),$$

where $Z_{n,m}(w)$ denotes the vector of length $2n$ each of whose first $n$ entries is $(-1)^{n-m+1}F_m(w)$ and each of whose last $n$ entries is $(-1)^{n-m}F_m(w)$. Let $w'$ be any one of the $m$ distinct zeros of $F_m(w)$. By (2.5),

$$M(2n, m, w')\tilde{Y}_{n,m}(w') = Z_{n,m}(w') = 0.$$

Thus the nonzero vector $\tilde{Y}_{n,m}(w')$ is in the nullspace of $M(2n, m, w')$, so the determinant of $M(2n, m, w')$ must be 0. Since the polynomial $D(w)$ vanishes at each zero of $F_m(w)$, it must be divisible by $F_m(w)$, and hence by $F_m(w)^2$. The polynomial $D(w)$ has degree $2m$, so $D(w) = \alpha F_m(w)^2$, where $\alpha$ is the leading coefficient of $D(w)$. It remains to prove that $\alpha = 1$. From the Leibniz formula for the determinant, it can be seen that $\alpha$ equals the determinant of the $(2n - 2m) \times (2n - 2m)$ submatrix situated in the center of the matrix $M(2n, m, w)$. This submatrix is $E(2n - 2m)$, so by Lemma 2.2, $\alpha = d(2n - 2m) = 1$. \qed
Theorem 2.4. Write $M = M(2n, m, w)$ and let $Pf(M)$ denote its Pfaffian. For each $w \in \mathbb{R}$ and all $n \geq 1$ with $n \geq m \geq 0$, we have

$$Pf(M) = (-1)^{(m+1)/2}F_m(w).$$

Proof. Let $\beta$ denote the leading coefficient of the polynomial $Pf(M)$. In view of Lemma 2.3, it suffices to show that

$$(2.6) \quad \beta = (-1)^{(m+1)/2}.$$

From [1, eq. 1.4], (2.6) holds when $n = 1$, so assume that $n > 1$. For $2 \leq j \leq 2n$, let $M'^{(j)}$ denote the $(2n - 2) \times (2n - 2)$ matrix obtained by removing the first and $j$-th rows and columns. From [1, eq. 1.5],

$$(2.7) \quad Pf(M) = \sum_{j=2}^{2n} (-1)^j a_j Pf(M'^{(j)}),$$

where $a_j$ is the $j$-th entry in the first row of $M$.

When $m = 0$, it follows by induction on $n$ that the right side of (2.7) is the alternating sum $1 - 1 + 1 - \cdots + 1$, so that $Pf(M) = 1$, proving (2.6). Thus assume that $m \geq 1$. The only term on the right side of (2.7) that contributes to the leading coefficient of $Pf(M)$ is the term for $j = 2n+1-m$, and for this $j$, $a_j = -w$. By induction on $m$, we may assume that for this $j$, $Pf(M'^{(j)}) = (-1)^{m/2}$. Therefore the right side of (2.7) contributes $(-1)^{m+[m/2]}$ to the leading coefficient of $Pf(M)$. Since $(-1)^{m+[m/2]} = (-1)^{(m+1)/2}$, the proof of (2.6) is complete. \hfill \Box

References


