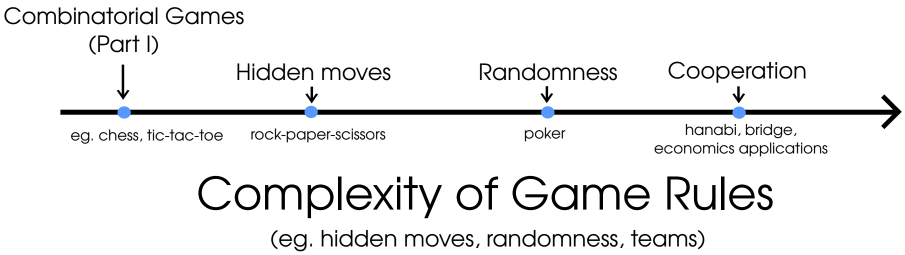
Math 152: Applicable Mathematics and Computing

April 3, 2017

Welcome to Math 152!

- This is "Applicable Mathematics and Computing". This time around, the topic will be an introduction to **Game Theory**.
- Textbook: *Game Theory* by Thomas S. Ferguson, available online.
- We begin by studying very simple games, and in many cases we can completely solve these games.
- As the course progresses, we will see more advanced games (eg. poker) as well as real-world applications.

Where this course is going, in a picture



Where this course is not going...

(Credit: TripAdvisor)

The 3/4 Guessing Game

Question (3/4 Game)

Consider a game with 10 players. Each player simultaneously guesses a real number between 0 and 100 (inclusive). The winner is the player that guesses closest to 3/4 of the average of the 10 guesses.

What is a good strategy for this game?

What about if we change the rules so that each player must guess an integer?

Course Logistics

- Homeworks due on Wednesdays (starting Week 2), in APM Basement. Lowest dropped.
- Two midterms (Weeks 3 and 7).
- Final score: 20% homework + 20% midterm1 + 20% midterm2 + 40% final.
- Full course details on webpage: `http://math.ucsd.edu/~rjtobin/152/`
- If you have any questions: send me an email! Either `rjtobin@ucsd.edu` or `rotobin@ucsd.edu`.

What is a Combinatorial Game?

A **combinatorial game** is a two-person turn-based game with perfect information, no chance moves and where the outcome is either a win or a loss (with a win for one player being a loss for the other player).

- turn-based: The players take their turns one after the other.
- perfect information: No hidden moves, or hidden objects in the game.
- no chance moves: There is no random component in the game.

Example: Simple Take-Away Game

Consider the following simple two-player game. Call the players “Player I” and “Player II”.

- There is a pile of n chips (or coins) on a table.
- A *move* consists of removing either one, two or three chips from the table.
- Players take their turns one after the other.
- The player that takes the last chip wins.

Let’s say there are 25 chips in the beginning, and Player I moves first. How do we answer these basic questions: What is a good move for Player I? Can Player I guarantee that they win?

We will show that a game like this can be analyzed simply using a technique called **backwards induction**.

Example: Simple Take-Away Game, Backwards Induction

In **backwards induction** we start by considering very small games (ie. very few chips), and then consider larger and larger games.

Let's say we are Player I, and we move first. Let's see if we can win this take-away game, for a given number of starting chips.

Number of chips	1	2	3	4	5	6	7	8	9	10	11	12
Winning player	I	I	I	II	I	I	I	II	I	I	I	II

Note: We make the assumption that our opponents will play optimally.

Definition of a Combinatorial Game

Def. A *combinatorial game* consists of the following

1. There are two players.
2. There is a set (usually finite) of possible positions in the game.
3. For each player and each position, there is a set of legal “moves” that this player can make. A move is just a new game position.
4. The players take turns moving.
5. The game ends when a player has no legal move on their turn. In **normal play** the last player to move wins. In **misère play**, the last player to move loses.

In some games, the available moves are identical for both players (eg. the take-away game). In others they are not (eg. chess). Games of the first type are called **impartial games**, and games of the second type are called **partizan games**.

Definition of a Combinatorial Game, Example

For the take-away game from earlier, the "positions" of the game are exactly the different numbers of chips that are currently in the pile:

$$\text{Positions} = \{0, 1, 2, 3, \cdots\}$$

Given some game position n, the set of legal moves is given by the new positions that can occur after we complete our turn. For example, the moves available at position 10 are given by:

$$\text{Moves}(10) = \{7, 8, 9\}$$

since if there are 10 chips, we can remove 1, 2 or 3 chips.
The moves available to Player I are the same as those available to Player II. So the game is **impartial**.

Combinatorial Games: The Restrictions

- One player wins and one player loses: no draws. This should exclude Chess and Tic-Tac-Toe. But we can get around this: consider a draw as a loss (or a win).
- No hidden moves. This exclude Rock-Paper-Scissors.
- No randomness. This excludes poker (and any card game that involves a shuffled deck).
- We will study more general games shortly.

P- and N-positions

- When solving the take-away game, we assumed we were Player **I** and it was our turn. These sorts of assumptions are ugly.
- More generally we will talk about game positions being **P-positions** or **N-positions**. **Informally**: a P-position is a game position where the **P**revious player that moved will win. An N-position is a game position where the **N**ext player that moves will win.
- For example, in the take-away game, the positions 1 is an N-position. The position 4 is a P-position. The position 5 is an N-position.

Number of chips	1	2	3	4	5	6	7	8	9	10	11	12
Winning player	**I**	**I**	**I**	**II**	**I**	**I**	**I**	**II**	**I**	**I**	**I**	**II**

Number of chips	1	2	3	4	5	6	7	8	9	10	11	12
Winning player	N	N	N	P	N	N	N	P	N	N	N	P

P- and N-positions: Definition

- **Def.** A *terminal position* if an impartial game is a position from which there are no legal moves.
- **Def.** *P-positions* and *N-positions* are defined recursively by the following three statements:
    1. All terminal positions are P-positions.
    2. From every N-position, there is at least one move to a P-position.
    3. From every P-position, every move is to an N-position.

P- and N-positions: Algorithm

1. Make a table with the different positions in the top row.
2. For each terminal position, label it as a P-position.
3. If there is an unlabelled position that has a move to *any* P-position, mark this position as an N-position.
4. If there is an unlabelled position where *every* move leads to an N-position, mark this position as a P-position.
5. Repeat steps 3 and 4 until all positions are labelled.

Board example. Consider a *subtraction game* where there is a pile of n coins, and in each turn a player may remove 1, 2 or 4 coins. The last player to remove a coin is the winner. Determine if the game with 20 chips is an N-position or a P-position.

P- and N-positions: Algorithm

Question (Subtraction game)

Consider the subtraction game where on each turn a player may remove 1, 3 or 4 coins, and the last player to remove a coin wins. Determine which player will win if there are 15 coins.

What if there are 1000 coins?