Math 152: Applicable Mathematics and Computing

April 24, 2017

Announcements

- Midterms results will be on TED before Friday (drop deadline).
- The final course grades will be calculated against a curve.
- Solutions to question 4 on the website (rest will be posted later).
- Josh's office hours at 1PM today delayed until 1.30PM.

Saddle Points

- Sometimes it is easy to find the value of the game from the payoff matrix.
- **Definition:** A saddle point is an entry a_{ij} of the matrix A such that
  (1) a_{ij} is the minimum number in the ith row
  (2) a_{ij} is the maximum number in the jth column
- If there is a saddle point, it is the value of the game. For example:

$$A = \begin{pmatrix} 4 & 1 & -3 \\ 3 & 2 & 5 \\ 0 & 1 & 6 \end{pmatrix}$$

- The optimal strategies are pure strategies: Player I picks the row containing the saddle point, Player II picks the column containing the saddle point.

Solution to 2 by 2 Matrix Games

We now have two ways to find optimal strategies for zero–sum games:

(1) find a saddle point

(2) find an equalizing strategy

In fact for a 2×2 matrix, if there is no saddle point, we are guaranteed to be able to find an equalizing strategy. So we can now solve 2×2 games: (1) first look for saddle points, if not (2) find equalizing strategy.

2 by 2 Solution Examples

- **Example 1.** $\begin{pmatrix} -2 & 3 \\ 3 & -4 \end{pmatrix}$
- **Example 2.** $\begin{pmatrix} 0 & -10 \\ 1 & 2 \end{pmatrix}$

Dominated Strategies

- A trick to help finding the optimal mixed strategy is to ignore any pure strategies that are obviously bad for one of the players.
- For example:

$$\begin{pmatrix} 5 & 2 & 3 \\ -1 & 7 & 8 \\ 0 & 1 & 2 \end{pmatrix}$$

- The Player I strategy corresponding to the bottom row is always worse for Player I that the top row. So let's just delete it.
- Similarly, the last column is always worse for Player II than the second last column. So we can delete it too.

$$\begin{pmatrix} 5 & 2 \\ -1 & 7 \end{pmatrix}$$

Dominated Strategies

- This trick is called removing dominated strategies.
- Row i of the matrix A is dominated by row k if $a_{ij} \leq a_{kj}$ for all j (and strictly dominated if $a_{ij} < a_{kj}$).
- Column j is dominated by column k if $a_{ij} \geq a_{ik}$ for all i.
- Removing dominated rows and columns does not change the value of the game.

  Remark: Sometimes there are several optimal strategies for a player. In this case, removing a dominated row or column can sometimes remove one of these optimal strategies. However, there is always at least one optimal strategy left, so *the value of the game does not change when we delete a dominated row or column.*

Dominated Strategies 2

- We have seen that if there is a row which is smaller than some other row (entrywise) then we can delete that row.
- Actually we can be even more general than this: if some row is smaller than any weighted average of the other rows, we can delete that too (similarly, if any column is bigger than any weighted average of the other columns, we can delete it too).
- For example:

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 5 & 2 \\ 4 & -4 & 1 \end{pmatrix}$$

- The top row is dominated by the average of the second two rows.

Latin Square Games

- A Latin square is an $n \times n$ array consisting of n different letters such that each letter appears exactly once in every row and column (a bit like Sudoku).
- If we assign a numerical value to each of these n letters, then the Latin square becomes a game.
- For example

$$\begin{pmatrix} a & b & c & d & e \\ b & e & a & c & d \\ c & a & d & e & b \\ d & c & e & b & a \\ e & d & b & a & c \end{pmatrix} \to \begin{pmatrix} 0 & 1 & -1 & 1 & 9 \\ 1 & 9 & 0 & -1 & 1 \\ -1 & 0 & 1 & 9 & 1 \\ 1 & -1 & 9 & 1 & 0 \\ 9 & 1 & 1 & 0 & -1 \end{pmatrix}$$

where $a = 0, b = 1, c = -1, d = 1, e = 9$.

Latin Square Games: Solution

- Games of this form have surprisingly simple solutions. In fact the optimal strategy for both players is to select a row/column uniformly–at–random (that is, randomly pick a row/column, where each row/column is equally likely to be picked).
- How do we prove that this is optimal? We just need to compute the expected winnings for each player using this strategy. (Remember: a pair of strategies is optimal if the max expected loss for II is the same as the min expected gain for I).

Latin Square Games: Solution

- Let $\mathbf{p} = \mathbf{q} = \left[\begin{array}{ccccc} 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \end{array}\right]^T$. Then:

$$\left[\begin{array}{ccccc} 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \end{array}\right] \begin{pmatrix} 0 & 1 & -1 & 1 & 9 \\ 1 & 9 & 0 & -1 & 1 \\ -1 & 0 & 1 & 9 & 1 \\ 1 & -1 & 9 & 1 & 0 \\ 9 & 1 & 1 & 0 & -1 \end{pmatrix}$$

$$= \left[\begin{array}{ccccc} 2 & 2 & 2 & 2 & 2 \end{array}\right]$$

- Similarly, $A\mathbf{q} = \left[\begin{array}{ccccc} 2 & 2 & 2 & 2 & 2 \end{array}\right]^T$.
- So the value of the game is 2 (and in fact, these are equalizing strategies, because the expected gain for each player is the same no matter what their opponent does).

Example

Find the value of the following game, and optimal strategies for both players.

$$\begin{pmatrix} 0 & 4 & 6 \\ 5 & 7 & 4 \\ 9 & 6 & 3 \end{pmatrix}$$

Example

In "Normandy: Game and Reality" by W. Drakert, an analysis is given of the Allied invasion of Normandy (D-Day). Six possible attacking configurations (1-6) for the Allies were given, and six possible defensive strategies (A-F) for the Germans. All 36 pairs were evaluated and simulated. The following table gives a numerical estimate of the value of each situation to the Allied forces.

$$\begin{array}{cc} & \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{pmatrix} 13 & 29 & 8 & 12 & 16 & 23 \\ 18 & 22 & 21 & 22 & 29 & 31 \\ 18 & 22 & 31 & 31 & 27 & 37 \\ 11 & 22 & 12 & 21 & 21 & 26 \\ 18 & 16 & 19 & 14 & 19 & 28 \\ 23 & 22 & 19 & 23 & 30 & 34 \end{pmatrix} \end{array}$$

Find the value of this game. The Allies and Germans opted for strategies 1 and B respectively, were these good choices?