Math 152: Applicable Mathematics and Computing

April 26, 2017

Announcements

- Homework 3 will be posted today (and due next Wednesday).
- Josh's office hours today start at 1.30PM.
- On Friday, Prof. Leonard Haff will give a guest lecture, "The Partizan Theory of Combinatorial Games. A Brief Introduction".

Graphical Method for Solving $2 \times m$ games

- When we found equalizing strategies for two–by–two games, we let p, $1 - p$ be the probabilities that Player 1 chose each of her strategies, and then computed the expectation for each of the possible strategies for Player II.
- We can do the same computations in $2 \times m$ games (so when there are two strategies for Player I), and by **plotting** the resulting functions we can find the optimal p.

Graphical Method Example

- Consider the game

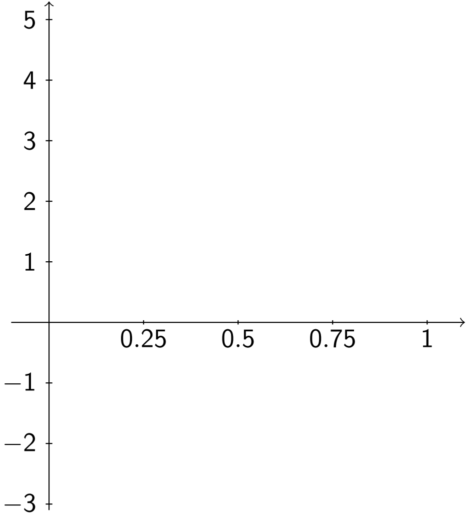
$$\begin{pmatrix} -3 & -1 & 3 & 4 \\ 5 & 1 & -2 & -1 \end{pmatrix}$$

- Let p be the probability that Player I picks the first row, and $1-p$ the probability that the second row is selected.
- Let's compute the expected gain for each column:

$$\begin{aligned} \mathbb{E}(\text{column 1}) &= -3p + 5(1-p) = -8p + 5 \\ \mathbb{E}(\text{column 2}) &= -1p + 1(1-p) = -2p + 1 \\ \mathbb{E}(\text{column 3}) &= 3p - 2(1-p) = 5p - 2 \\ \mathbb{E}(\text{column 4}) &= 4p - 1(1-p) = 5p - 1 \end{aligned}$$

Graphical Method Example

Plot each of these expectations as a function of p (Y-axis is expected payoff):



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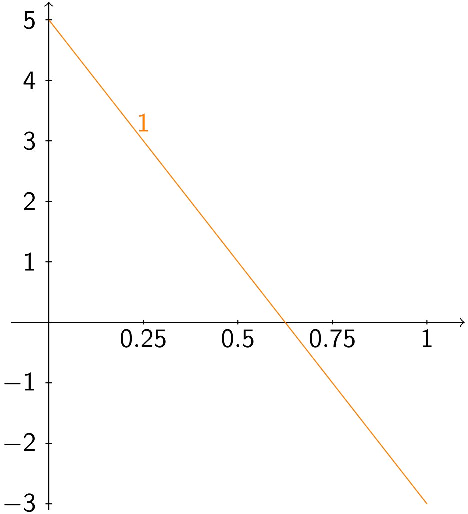

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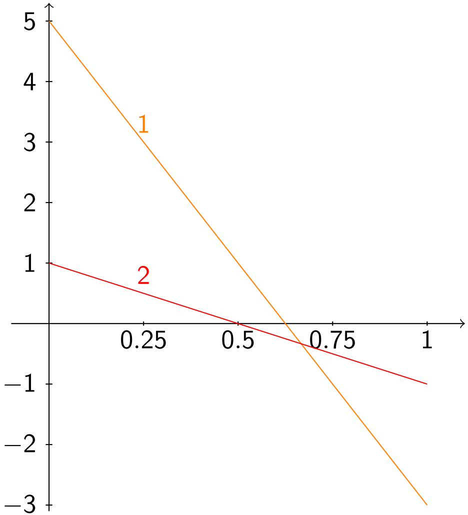

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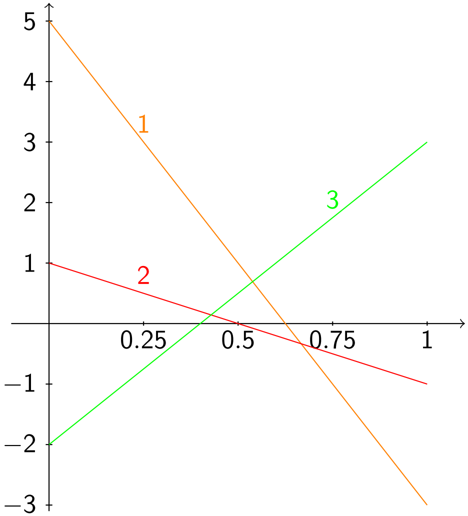

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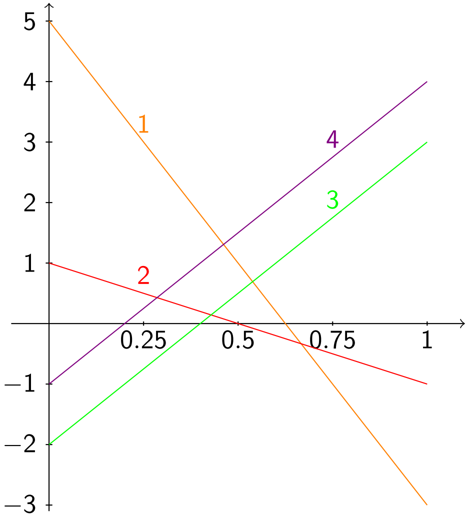

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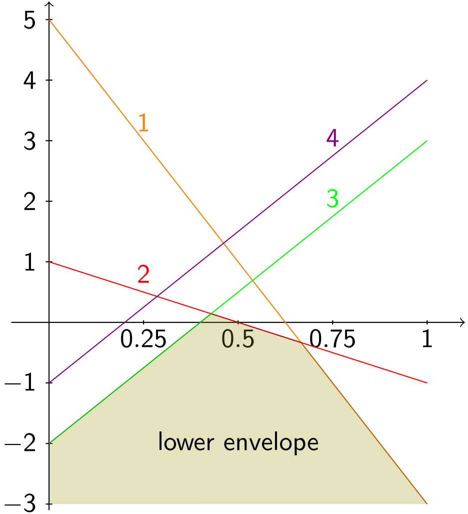

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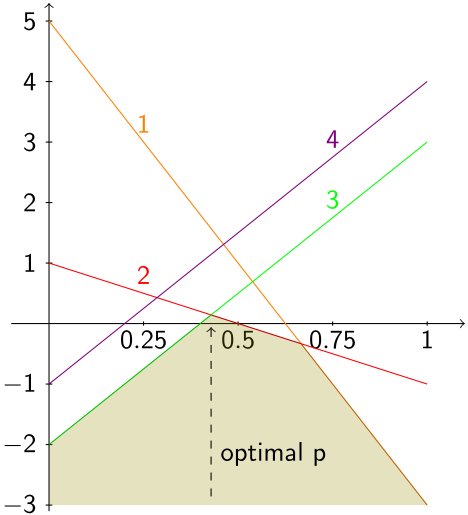

Solving $m \times 2$ games

Similarly, we can solve games that are $m \times 2$. For example:

$$\begin{pmatrix} -3 & 5 \\ -1 & 1 \\ 3 & -2 \\ 4 & -1 \end{pmatrix}$$

In this case, Player II has to choose one of the two columns. Player II wants the result to be **small**, so instead we look at the **lowest** point of the **upper** envelope.

Linear Algebra Review

- Recall that if Player I uses the mixed strategy **p**, and if Player II selects column j, then Player I's expected gain is the jth entry of the vector $p^T A$:

$$\sum_{i=1}^{m} p_i a_{ij}$$

- When a player uses her optimal mixed strategy, her expected gain is at least V no matter what j Player II chooses. So

$$\sum_{i=1}^{m} p_i a_{ij} \geq V, \quad \forall j$$

- Similarly, for Player II's optimal strategy **q**

$$\sum_{j=1}^{n} a_{ij} q_j \leq V, \quad \forall i$$

Linear Algebra Warm–Up

Lemma

If both players use their optimal strategies, **p** and **q**, then

$$\mathbf{p}^T A\mathbf{q} = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i a_{ij} q_j = V$$

Proof.

$$V = \sum_{j=1}^{n} V q_j$$

Linear Algebra Warm–Up

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Proof.

$$V = \sum_{j=1}^{n} V q_j \leq \sum_{j=1}^{n}\left(\sum_{i=1}^{m} p_i a_{ij}\right) q_j$$

□

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Proof.

$$V = \sum_{j=1}^{n} V q_j \leq \sum_{j=1}^{n}\left(\sum_{i=1}^{m} p_i a_{ij}\right) q_j \ = \sum_{j=1}^{n}\sum_{i=1}^{m} p_i a_{ij} q_j$$

$\square$

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If both players use their optimal strategies, **p** and **q**, then

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Proof.

$$V = \sum_{j=1}^{n} V q_j \leq \sum_{j=1}^{n}\left(\sum_{i=1}^{m} p_i a_{ij}\right) q_j = \sum_{j=1}^{n}\sum_{i=1}^{m} p_i a_{ij} q_j$$
$$= \sum_{i=1}^{m} p_i \left(\sum_{j=1}^{n} a_{ij} q_j\right)$$

□

Linear Algebra Warm–Up

Lemma

If both players use their optimal strategies, **p** and **q**, then

$$\mathbf{p}^T A\mathbf{q} = \sum_{i=1}^{m}\sum_{j=1}^{n} p_i a_{ij} q_j = V$$

Proof.

$$\begin{aligned} V = \sum_{j=1}^{n} V q_j \leq \sum_{j=1}^{n}\left(\sum_{i=1}^{m} p_i a_{ij}\right) q_j &= \sum_{j=1}^{n}\sum_{i=1}^{m} p_i a_{ij} q_j \\ &= \sum_{i=1}^{m} p_i \left(\sum_{j=1}^{n} a_{ij} q_j\right) \leq \sum_{i=1}^{m} p_i V = V \quad \square \end{aligned}$$

$\square$

Principle of Indifference

Theorem

Principle of Indifference Consider an $m \times n$ game A with value V, and optimal strategies $\mathbf{p}, \mathbf{q}$. Then

$$\sum_{j=1}^{n} a_{ij} q_j = V$$

for all i where $p_i > 0$. And Then

$$\sum_{i=1}^{m} p_i a_{ij} = V$$

for all j where $q_j > m$.

Principle of Indifference

This is saying: if we restrict our attention to only those columns that Player II's optimal strategies actually uses, then in those columns we should search for an equalizing strategy for Player I.

Similarly, if we restrict our attention to only those rows that Player I's optimal strategies actually uses, then in those columns we should search for an equalizing strategy for Player II.

Even or Odd Revisited

Game (Even/Odd 2)

Consider a game where both players simultaneously announce a number between 0 and 2 (inclusive). If the sum of the numbers is odd, then Player I wins. If the sum is even, the Player II wins. The payoff is the sum of the numbers.

Let's say we know that Player II's strategy will assign positive probability to each column. Solve the game using the Principle of Indifference.