Math 152: Applicable Mathematics and Computing

May 1, 2017

Announcements

- Josh's office hours today are 12.30–1.30 and 2.30–3.30.
- Homework 2 results posted on TED.
- If anyone cannot view their homework or midterm scores on TED, let me know.
- Midterm scores posted: score based on max of $0.25(Q1 + Q2 + Q3 + Q4)$ and $0.3(Q1 + Q2 + Q3) + 0.1(Q4)$.

90–100	21
80–89	40
70–79	58
60–69	38
50–59	17
0–49	8

Recap: Principle of Indifference

Last time we saw the Principle of Indifference. If **p** is an optimal strategy for Player I, and **q** is an optimal strategy for Player II, then the Principle of Indifference says:

- If $p_i > 0$ for some i, then if Player II uses strategy **q** and Player I selects row i, then the expected winnings for Player I are exactly V.
- If $q_j > 0$ for some j, then if Player I uses strategy **p** and Player II selects column j, then the expected winnings for Player I are exactly V.

Consequence: If **p** is strictly bigger than 0 in every entry, then **q** is an equalizing strategy for Player II. If **q** is strictly bigger than 0 in every entry, then **p** is an equalizing strategy for Player I.

Nonsingular Game Matrices

Theorem (Nonsingular Game Matrices)

Assume A is a nonsingular square matrix and $\mathbf{1}^T A^{-1}\mathbf{1} \neq 0$. Then the game with matrix A has value $V = (\mathbf{1}^T A^{-1}\mathbf{1})^{-1}$ and optimal strategies $\mathbf{p}^T = V\mathbf{1}^T A^{-1}$ and $\mathbf{q} = VA^{-1}\mathbf{1}$, provided that $\mathbf{p} \geq 0$ and $\mathbf{q} \geq 0$.

(The vector $\mathbf{1}$ is the vector with every entry equal to 1.)

Nonsingular Game Matrices: Proof

Proof.

Let $\mathbf{p}, \mathbf{q}$ be optimal strategies for Player I and II respectively. Assume that every entry of $\mathbf{p}$ is strictly positive.
As observed earlier, this means that $\mathbf{q}$ is an equalizing strategy. So

$$A\mathbf{q} = V\mathbf{1}$$

So $\mathbf{q} = VA^{-1}\mathbf{1}$. We know that $\mathbf{1}^T\mathbf{q} = 1$, so

$$1 = V\mathbf{1}^T A^{-1}\mathbf{1} \Rightarrow V = (\mathbf{1}^T A^{-1}\mathbf{1})^{-1}$$

Repeating this for $\mathbf{p}$ gives $\mathbf{p}^T = V\mathbf{1}^T A^{-1}$.
If both $\mathbf{p} \geq 0$ and $\mathbf{q} \geq 0$, then we have a pair of optimal strategies. If not, these are not valid strategies and so our assumption at the beginning was wrong. □

Nonsingular Game Matrices: Example 1

Theorem (Nonsingular Game Matrices)

Assume A is a nonsingular square matrix and $\mathbf{1}^T A^{-1}\mathbf{1} \neq 0$. Then the game with matrix A has value $V = (\mathbf{1}^T A^{-1}\mathbf{1})^{-1}$ and optimal strategies $\mathbf{p}^T = V\mathbf{1}^T A^{-1}$ and $\mathbf{q} = VA^{-1}\mathbf{1}$, provided that $\mathbf{p} \geq 0$ and $\mathbf{q} \geq 0$.

Example. Solve the game

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & 4 \\ -1 & 4 & -3 \end{pmatrix}$$

Note that

$$A^{-1} = \frac{1}{16}\begin{pmatrix} 13 & -2 & -7 \\ -2 & 4 & 6 \\ -7 & 6 & 5 \end{pmatrix}$$

Nonsingular Game Matrices: Example 2

Theorem (Nonsingular Game Matrices)

Assume A is a nonsingular square matrix and $\mathbf{1}^T A^{-1}\mathbf{1} \neq 0$. Then the game with matrix A has value $V = (\mathbf{1}^T A^{-1}\mathbf{1})^{-1}$ and optimal strategies $\mathbf{p}^T = V\mathbf{1}^T A^{-1}$ and $\mathbf{q} = VA^{-1}\mathbf{1}$, provided that $\mathbf{p} \geq 0$ and $\mathbf{q} \geq 0$.

Example. Solve the game

$$B = \begin{pmatrix} 8 & 16 & -8 \\ 16 & 17 & 2 \\ -8 & 2 & -4 \end{pmatrix}$$

Note that

$$B^{-1} = \frac{1}{48}\begin{pmatrix} 3 & -2 & -7 \\ -2 & 4 & 6 \\ -7 & 6 & 5 \end{pmatrix}$$

Diagonal Matrices

Recall that a diagonal matrix is a square matrix where all entries not on the main diagonal are 0. We can use the previous theorem to solve any diagonal matrix with positive diagonal entries.

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

Triangular Matrices

A triangular matrix is a square matrix where all entries except the entries *below* the main diagonal are 0. Using the Principle of Indifference, we can often solve games like this (this does not always work though!).

Question

Using the Principle of Indifference, solve the game

$$A = \begin{pmatrix} 1 & -2 & 3 & -4 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Symmetric Games

Def. A matrix A is skew-symmetric if $A^T = -A$.

Def. A finite game is symmetric if the underlying payoff matrix is square and skew-symmetric.

Theorem

A finite symmetric game has value zero, and any optimal strategy for one player is also optimal for the other player.

Game (Rock Paper Scissors)

Consider rock-paper-scissors where the winner receives payoff 1, and in the case of a draw the payoff is 0.