Math 152: Applicable Mathematics and Computing

May 3, 2017

Announcements

- If you cannot see your Midterm result on TED, make sure you are choosing the ‘All’ option in the Grade Center.
- Homework 3 due today.

Invariance

Game (Matching Pennies)

Both players simultaneously choose either ‘heads’ or ‘tails’. If the two choices are the same, then Player I wins 1. If they are different, Player II wins 1.

The payoff matrix for this game is:

$$\begin{array}{c c} & \begin{array}{cc} H & T \end{array} \\ \begin{array}{c} H \\ T \end{array} & \begin{pmatrix} +1 & -1 \\ -1 & +1 \end{pmatrix} \end{array}$$

Notice that if we swap ‘H’ and ‘T’ in this matrix, the payoff matrix remains the same:

$$\begin{array}{c c} & \begin{array}{cc} T & H \end{array} \\ \begin{array}{c} T \\ H \end{array} & \begin{pmatrix} +1 & -1 \\ -1 & +1 \end{pmatrix} \end{array}$$

Invariance

- Reordering the rows and columns of the payoff matrix does not change the rules of the game, but usually this will change the matrix itself.
- **Def.** A matrix game is invariant under some reordering of rows and columns if the payoff matrix is the same after we perform the reordering.
- For example, the *Matching Pennies* game was invariant under the reordering: 'H' $\to$ 'T', 'T' $\to$ 'H'.
- And *Rock-Paper-Scissors* is invariant under the reordering: 'R' $\to$ 'P', 'P' $\to$ 'S', 'S' $\to$ 'R'.

Invariance

Theorem

Let A be a matrix game that is invariant under some reordering.

- If x, y are two rows, where $x \to y$ under the reordering, then there is an optimal strategy for Player I with $\mathbf{p}(x) = \mathbf{p}(y)$.
- If x, y are two columns, where $x \to y$ under the reordering, then there is an optimal strategy for Player II with $\mathbf{q}(x) = \mathbf{q}(y)$.

For example, use this to solve *Matching Pennies*:

$$\begin{array}{c c} & \begin{array}{cc} H & T \end{array} \\ \begin{array}{c} H \\ T \end{array} & \begin{pmatrix} +1 & -1 \\ -1 & +1 \end{pmatrix} \end{array}$$

Invariance Example

Question

Consider the following game:

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix} \end{array}$$

Solve this game using Invariance.

Note About Invariance

Note: In the textbook, the section on Invariance (Part II, 3.6) relies heavily on group theory (in particular, group actions). Since MA100 is not a prerequisite, I will not assume any knowledge of this. For section 3.6, all that you need to know is what is contained in these slides.

Best Response

- For this section we will investigate what happens if a player knows what (mixed) strategy their opponent is going to use.
- First we establish some notation. Remember that X and Y are the sets of pure strategies for Player I and Player II respectively.
- Let X^* and Y^* be the set of all mixed strategies for Player I and Player II respectively.

Best Response

- Assume that Player I knows that Player II will use strategy $\mathbf{q} \in Y^*$. Then Player I should choose a strategy $\mathbf{p}$ that maximizes their average winnings.
- **Def.** Given a (mixed) strategy $\mathbf{q}$ for Player II, the best response strategy for Player I is the strategy $\mathbf{p}$ that maximizes $\mathbf{p}^T A\mathbf{q}$.
- **Def.** Given a (mixed) strategy $\mathbf{p}$ for Player I, the best response strategy for Player II is the strategy $\mathbf{q}$ that minimizes $\mathbf{p}^T A\mathbf{q}$.

Question

Given Player II's strategy is $\mathbf{q} = [0.5\ 0.2\ 0.3]^T$, find the best response strategy for Player I, for the game below.

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & -2 & 3 \\ 2 & 3 & -4 \end{pmatrix}$$

Best Response: Upper Value

- As in the previous slide, assume that Player II must announce their strategy $\mathbf{q}$ before Player I chooses their strategy.
- For a finite game, the best reponse strategy can always be taken to be a pure strategy: Player I will choose whichever row gives them the buggest average payoff (this is a pure strategy).
- Player II wants to choose $\mathbf{q}$ so that Player I's best response payoff is as small as possible. Let $\overline{V}$ be this minimum payoff. It is called the upper value of the game.

Best Response: Upper Value

- From the previous slide, $\overline{V}$ is the least amount that Player II will lose, on average, if they must announce their strategy before Player I chooses theirs.
- Written mathematically, Player II must solve:

$$\overline{V} = \min_{\mathbf{q} \in Y^*} \max_{1 \le i \le m} \sum_{j=1}^{n} a_{ij} q_j = \min_{\mathbf{q} \in Y^*} \max_{\mathbf{p} \in X^*} \mathbf{p}^T A \mathbf{q}$$

- The strategy $\mathbf{q}$ that attains this minimum is called the minimax strategy for Player II. It is the **same** as the optimal strategy we defined before.
- Since announcing their strategy first cannot be an advantage, we have $V \le \overline{V}$.

Best Response: Lower Value

- Similarly, $\underline{V}$ is the least amount that Player I will win, on average, if they must announce their strategy before Player II chooses theirs.
- Written mathematically, Player I must solve:

$$\underline{V} = \max_{\mathbf{p} \in X^*} \min_{1 \le j \le n} \sum_{i=1}^{m} p_i a_{ij} = \max_{\mathbf{p} \in X^*} \min_{\mathbf{q} \in Y^*} \mathbf{p}^T A \mathbf{q}$$

- $\underline{V}$ is the lower value of the game, and the **p** attaining the max will be the same as the optimal strategy we found before.
- Since announcing their strategy first cannot be an advantage, we have $\underline{V} \le V \le \overline{V}$.