Math 152: Applicable Mathematics and Computing

May 5, 2017

Upper and Lower Values of Games (tl;dr version)

- The upper value of a game $\overline{V}$ is the value of a game when Player II is forced to announce their strategy first.

$$V \leq \overline{V}.$$

- The lower value of a game $\underline{V}$ is the value of a game when Player I is forced to announce their strategy first.

$$\underline{V} \leq V.$$

- We will see a full example of this now.
- **Remark.** For finite games, in fact we have $\underline{V} = V = \overline{V}$.

Best Response

Question

Consider the game

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$$

Say Player I announces that they are using strategy $\mathbf{p} = (p \;\; 1 - p)$.

(a) Find Player II's best response strategy.

(b) What is the optimal choice of p for Player I? Hence find the lower value of the game.

Best Response

(a) Find Player II's best response strategy.
First we find the expected payoff for each of the two columns that Player II can choose.

$$\mathbf{p}^T A = \begin{pmatrix} 2p & -2p+3 \end{pmatrix}$$

Player II will choose whichever column is smaller. So they choose column 1 if

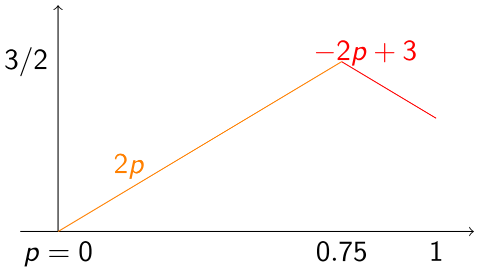
$$2p < -2p + 3 \Leftrightarrow p < 3/4.$$

And they will choose column 2 if

$$2p \geq -2p + 3 \Leftrightarrow p \geq 3/4.$$

Best Response

(b) What is the optimal choice of p for Player I? Hence find the lower value of the game.



$p = 0.75$ gives the highest payoff, and so this is what they will choose. In this situation, where Player I announces their strategy in advance, the expected payoff for Player I is: at least

$$\underline{V} = 2(0.75) = 1.5$$

Fictitious Play

- Say we are playing a game repeatedly. We would like to know what strategy our opponent is going to use. How can we guess?
- A natural approach is to estimate that the probability our opponent will some pure strategy is the proportion of times they have used that strategy before.
- This estimate gives an **upper bound** on our expected winnings. It is only a bound, since maybe we are wrong about our opponent's strategy, and so maybe they can do better.

Fictitious Play

Question

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -3 \\ -1 & 2 & 2 \end{pmatrix}$$

We are Player I. In the past five games, our opponent chose columns: 3, 1, 3, 2, 1.

(a) What is our estimate of our opponent's mixed strategy?

(b) Which row should we choose in the next game?

(c) Using (b), get an upper bound on the value of the game.

Fictitious Play

(a) What is our estimate of our opponent's mixed strategy?

Well, they have used column 1 twice, out of a total of five games. So our estimate for their probability of picking column 1 is $2/5$.

They have used column 2 once, so our estimate for that column is $1/5$.

Finally they have used column 3 twice, so our estimate for that column is $2/5$.

In total, our estimate for their strategy is $\mathbf{q}_{\text{guess}} = (2/5 \;\; 1/5 \;\; 2/5)^T$.

Fictitious Play

(b) Which row should we choose in the next game? From the previous slide, we have $\mathbf{q}_{\text{guess}} = (2/5 \;\; 1/5 \;\; 2/5)^T$. So the expected payoffs are

$$A\mathbf{q}_{\text{guess}} = \begin{pmatrix} 1 \\ -1 \\ 0.8 \end{pmatrix}$$

So we should choose row 1.

Remark: We can compute this a little faster. Just add each of the columns that our opponent has selected so far (columns 3, 1, 3, 2 and 1), and pick the row with the biggest total.

Fictitious Play

(c) Using (b), get an upper bound on the value of the game. We have decided to choose row 1. If we are correct about our opponent's strategy, then our payoff is 1.

However maybe we are wrong about our opponent's strategy. Their optimal strategy might be better for them. So we might win less than 1. So we have learned that $V \leq 1$.

Remark: If we were Player II, we would get a **lower bound** for the value of the game instead.

Fictitious Play

- So now we (finally!) get to the point of what we have been talking about the past few days.
- We have discussed:
  1. If we know what strategy our opponent will use, we know how to play (*best response*).
  2. When playing repeatedly, Player I can estimate Player II's strategy, use best response, and get an **upper bound** for the value of the game.
  3. When playing repeatedly, Player II can estimate Player I's strategy, use best response, and get a **lower bound** for the value of the game.

Putting these together, we can play the game repeatedly, and get upper and lower bounds on the value of the game. Play long enough and we find the true value. This technique is called fictitious play, and we can use it to estimate the value of very large games.

Fictitious Play

Algorithm (Fictitious Play)

1. Player I chooses the first row.
2. Player II chooses a best response based on all of Player I's previous choices. This gives a lower bound for V.
3. Player I chooses a best response based on all of Player II's previous choices. This gives an upper bound for V.
4. Repeat step 2, until we are satisfied with our upper and lower bounds.

Fictitious Play

Question

$$A = \begin{pmatrix} 2 & -1 & 6 \\ 0 & 1 & -1 \\ -2 & 2 & 1 \end{pmatrix}$$

Using fictitious play, estimate the value of this game to within an error of at most 0.5.

Fictitious Play

(1) Player I starts by choosing row 1.

(2) Player II looks at Player I's choices so far: row 1. This is

$$\begin{pmatrix} 2 & -1 & 6 \end{pmatrix}$$

So Player II chooses column 2. The expected payoff is -1. This is a lower bound on V.

(3) Player I looks at Player II's choices so far: column 2. This is

$$\begin{pmatrix} -1 & 1 & 2 \end{pmatrix}^T$$

So Player I chooses row 3. The expected payoff is 2. This is an upper bound on V.

Fictitious Play

(4) Player II looks at Player I's choices so far: row 1, row 3. The sum of these is

$$\begin{pmatrix} 0 & 1 & 7 \end{pmatrix}$$

So Player II chooses column 1. The expected payoff is $0/2 = 0$. (The "2" here comes from the fact that the 0 is the payoff for two rounds, so the average payoff each round was $0/2$). This is a lower bound on V.

(5) Player I looks at Player II's choices so far: column 1, column 2. This is

$$\begin{pmatrix} 1 & 1 & 0 \end{pmatrix}^T$$

So Player I chooses row 1. The expected payoff is $1/2$. This is an upper bound on V.

So: $0 \le V \le 1/2$.

Even or Odd Revisited (Again)

Game (Even/Odd 100)

Consider a game where both players simultaneously announce a number between 1 and 100 (inclusive). If the sum of the numbers is odd, then Player I wins. If the sum is even, the Player II wins. The payoff is the sum of the numbers.

Using the method of fictitious play (and a computer) find the value of this game and a pair of optimal strategies.