Math 152: Applicable Mathematics and Computing

May 8, 2017

Extensive Form: Overview

- We have been studying the strategic form of a game: we considered only a player's overall strategy, and not what happens on a turn-by-turn basis.
- Today we will see the extensive form of a game, which examines a game in much more detail.
- We will see how to go back and forth between these two forms of a game.

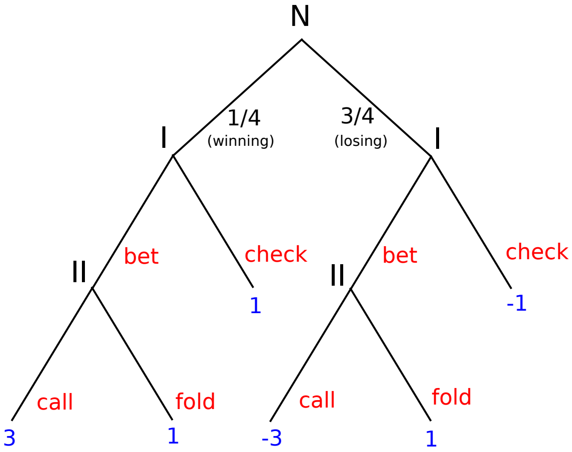
Motivating Example

Game (Poker Endgame)

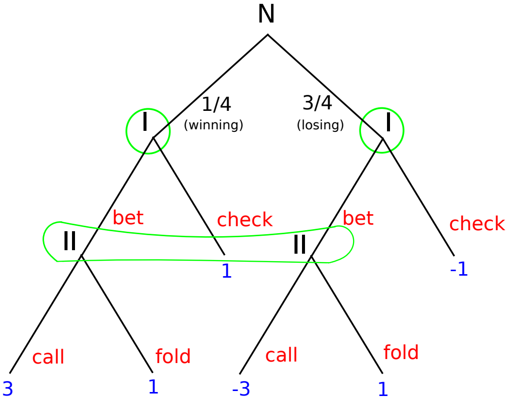
Two players are playing Poker, and the game is nearly over. There is 1 dollar currently at stake.

1. Player I will be given one final card from the dealer, which Player II will not see. With probability $1/4$ this card is a winning card for Player I, and with probability $3/4$ it is a losing card.
2. Player I can either bet an additional 2 dollars or check. If they choose to check, they reveal their card and the winner receives 1 dollar.
3. If Player I chose to bet, then Player II must either call or fold. If Player II chooses call, then Player I reveals their card and the winner receives 3 dollars from the loser. If Player II chooses fold, Player II loses and Player I receives 1 dollar.

Motivating Example



Motivating Example



Note: Player II does not know what card Player I receives in the first step. So from Player II's perspective, the two different vertices 'II' are identical. We say that these two positions are in the same information set.

Trees

- Recall that we defined a directed graph as a set of vertices X and a function F that for each vertex x defines a set of followers $F(x)$. We draw this as an arrow from x to each vertex in $F(x)$.
- Also recall that a path is a sequence of vertices $x_1, x_2, \cdots, x_k$ where x_{i+1} is a follower of x_i.
- **Def.** A tree is a directed graph (T, F) in which there is a special vertex, t_0, called the root, such that for every other vertex t there is a *unique* path from t_0 to t.
- **Def.** If a vertex has no followers (ie. $F(x) = \emptyset$) then x is a terminal vertex.
- **Convention.** We will draw trees so that for any vertex x, all of the followers of x are drawn below x. This way, we do not need to draw the direction of any arrow.

Extensive Form

Def. A finite two-person zero sum game in extensive form is given by

(1) A finite tree with vertices T.

(2) A payoff function that assigns a real number to each terminal vertex.

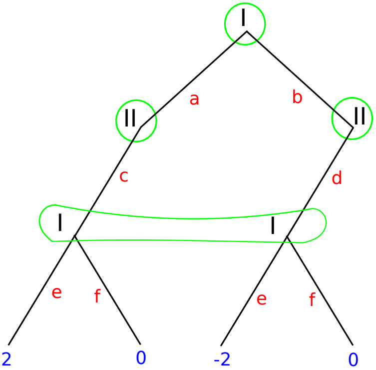
(3) A set T_0 of non-terminal vertices (representing positions at which chance moves occur) and for each $t \in T_0$, a probability distribution on the edges leading from t.

(4) For the rest of the vertices (ie. not terminal and not in T_0), a partition into two groups of information sets: $T_{11}, T_{12}, \cdots, T_{1k_1}$ (for Player I) and $T_{21}, T_{22}, \cdots, T_{2k_2}$ (for Player II).

(5) For each information set T_{jk}, a set of labels L_{jk} and for each $t \in T_{jk}$ a one-to-one mapping from L_{jk} to the edges leading from t.

Perfect Recall

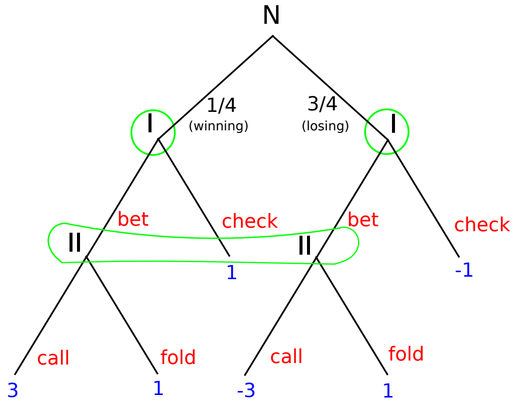
Note: We do not require that players remember all of their previous moves. For example, the game below can only arise if Player I forgets their first move. We can tell if a player remembers their moves from the information sets. If they do, it is a game of perfect recall.



From Extensive Form to Strategic Form

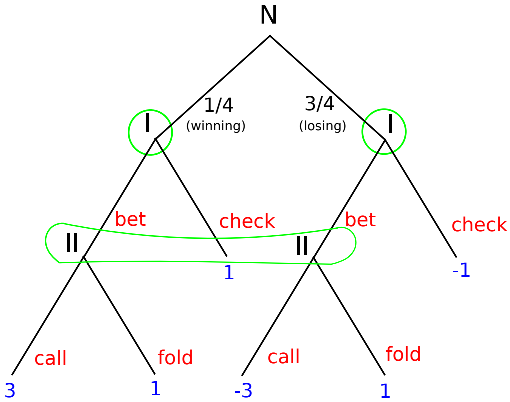
- Given the extensive form of a game, what are the pure strategies for each player?
- For every one of the player's vertices, that player needs to choose which follower to choose. Additionally, for two vertices in the same information set, the choice must be the same.
- So a strategy for a player is given by a choice of follower for each information set.

From Extensive Form to Strategic Form: Example



Player II has one information set, with two followers. So there are two pure strategies: c (call) and f (fold).

From Extensive Form to Strategic Form: Example



Player I has two information sets, each with two followers. So there are four pure strategies: (b_w, b_l), (b_w, c_l), (c_w, b_l), (c_w, c_l).

From Extensive Form to Strategic Form: Example

- So Player I has 4 pure strategies, and Player II has 2 pure strategies. Hence the payoff matrix of the strategic form will be a 4×2 matrix.
- For each pair of strategies, we compute the average payoff. Eg. for the strategies (b_w, b_l) and c: with probability $1/4$, the payoff is 3. With probability $3/4$, the payoff is -3. So the average payoff is:

$$A((b_w, b_l), c) = 3(1/4) - 3(3/4) = -6/4 = -3/2$$

- Or for the strategies (b_w, c_l) and c, the average payoff is

$$A((b_w, c_l), c) = 3(1/4) - 1(3/4) = 0$$

From Extensive Form to Strategic Form: Example

Filling in the full payoff matrix, we get:

$$\begin{array}{cc} & \begin{array}{cc} c & f \end{array} \\ \begin{array}{c} (b_w, b_l) \\ (b_w, c_l) \\ (c_w, b_l) \\ (c_w, c_l) \end{array} & \begin{pmatrix} -3/2 & 1 \\ 0 & -1/2 \\ -2 & 1 \\ -1/2 & -1/2 \end{pmatrix} \end{array}$$

What is the value of this game, and what are the optimal strategies?

General Poker Endgame

Question

In the Poker Endgame scenario above, Player I was allowed to bet an additional 3 dollars. More generally, say that they can bet an additional x dollars for some x. What should they choose x to be in order to maximize their winnings?

General Poker Endgame

In this case, the payoff matrix becomes:

$$\begin{array}{cc} & \begin{array}{cc} c & f \end{array} \\ \begin{array}{c} (b_w, b_l) \\ (b_w, c_l) \\ (c_w, b_l) \\ (c_w, c_l) \end{array} & \begin{pmatrix} \frac{-1-x}{2} & 1 \\ \frac{x-2}{4} & -1/2 \\ \frac{-2-3x}{4} & 1 \\ -1/2 & -1/2 \end{pmatrix} \end{array}$$

General Poker Endgame

In this case, the payoff matrix becomes:

$$\begin{array}{c c} & \begin{array}{cc} c & f \end{array} \\ \begin{array}{c} (b_w, b_l) \\ (b_w, c_l) \\ (c_w, b_l) \\ (c_w, c_l) \end{array} & \begin{pmatrix} \frac{-1-x}{2} & 1 \\ \frac{x-2}{4} & -1/2 \\ \frac{-2-3x}{4} & 1 \\ -1/2 & -1/2 \end{pmatrix} \end{array}$$

We can dominate the third row by the first, and the fourth by the second. That leaves:

$$\begin{array}{c c} & \begin{array}{cc} c & f \end{array} \\ \begin{array}{c} (b_w, b_l) \\ (b_w, c_l) \end{array} & \begin{pmatrix} \frac{-1-x}{2} & 1 \\ \frac{x-2}{4} & -1/2 \end{pmatrix} \end{array}$$