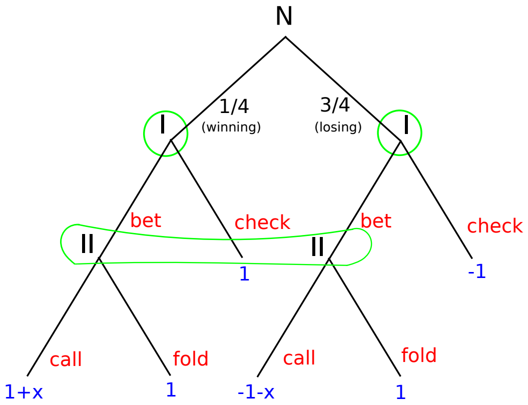
Math 152: Applicable Mathematics and Computing

May 10, 2017

Last Time



Last time: We saw that the value of this game is given by $V = -1/(x+2)$. So Player I should bet **arbitrarily large amounts**, which does not seem reasonable.

St. Petersburg Paradox

Game

Player II places 2 dollars in the pot. Then Player II repeatedly flips a coin. If the coin is *heads* then Player II doubles the amount of money in the pot. If the coin is *tails*, then Player I wins everything in the pot and the game ends.

This game is clearly biased towards Player I. How much should Player I be willing to pay in order to play the game?

St. Petersburg Paradox

Game

Player II places 2 dollars in the pot. Then Player II repeatedly flips a coin. If the coin is *heads* then Player II doubles the amount of money in the pot. If the coin is *tails*, then Player I wins everything in the pot and the game ends.

A natural approach is to compute the expected gain for Player I.

$$\begin{aligned} \mathbb{E}(\text{gain for I}) &= \left(\frac{1}{2}\right)2 + \left(\frac{1}{4}\right)4 + \left(\frac{1}{8}\right)8 + \cdots \\ &= 1 + 1 + 1 + \cdots \\ &= \infty \end{aligned}$$

This would suggest that it is worth paying *any* amount of money for the chance to play this game. This does not seem quite right.

St. Petersburg Paradox: Issues

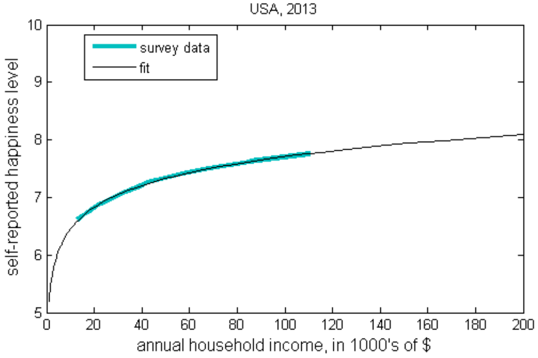
There are several issues with the analysis above, partly due to the use of *expectation*.

The following two scenarios have equal expected gain. Are they equally desirable?

1. You receive 100,000 dollars.
2. You flip a fair coin, and if it is heads you win 200,000 dollars but if it is tails you receive nothing.

St. Petersburg Paradox: Issues

Generally people prefer option 1: the guarantee of 100,000 dollars. Why? One reason is that 200,000 dollars will not make us twice as happy as 100,000 dollars. *Diminishing returns* are at play.



(Image: gravityandbrevity.wordpress.com, Data: Stevenson, Wolfers)

St. Petersburg Paradox: Issues

We can solve this by changing units. Instead of dollars, first convert to units where 200,000 really is twice as desirable as 100,000.

Eg. 100,000 units = 100,000 dollars, and 200,000 units = 1,000,000 dollars. Then our choices are:

1. You receive 100,000 dollars.
2. You flip a fair coin, and if it is heads you win 1,000,000 dollars but if it is tails you receive nothing.

Now maybe both options are equally desirable.

These new units are referred to as a utility function.

St. Petersburg Paradox: Issues

- Another issue with our analysis: it depends on Player II paying out huge sums of money.
- In reality, we know that there is some upper bound on how much they will pay.
- For example, if we know they have at most 1 million dollars, then our expected gain is only about 20 dollars.

Arbitrary Betting Poker Endgame

Game (Arbitrary Betting Poker Endgame)

Two players are playing Poker, and the game is nearly over. There is 1 dollar currently at stake.

1. Player I will be given one final card from the dealer, which Player II will not see. With probability 1/4 this card is a winning card for Player I, and with probability 3/4 it is a losing card.
2. Player I can either bet an additional x dollars or check, for any x between 1 and 100 (which they choose). If they choose to check, they reveal their card and the winner receives 1 dollar.
3. If Player I chose to bet, then Player II must either call or fold. If Player II chooses call, then Player I reveals their card and the winner receives $1 + x$ dollars from the loser. If Player II chooses fold, Player II loses and Player I receives 1 dollar.

Arbitrary Betting Poker Endgame

- What do strategies look like for this game?
- One type of strategy for Player I is: bet x if dealt a winning card, bet y otherwise (denote this by (b_x, b_y)).
- One type of strategy for Player II is: call if Player I bets less than z, fold otherwise (denote this by c_z).
- The expected payoff is:

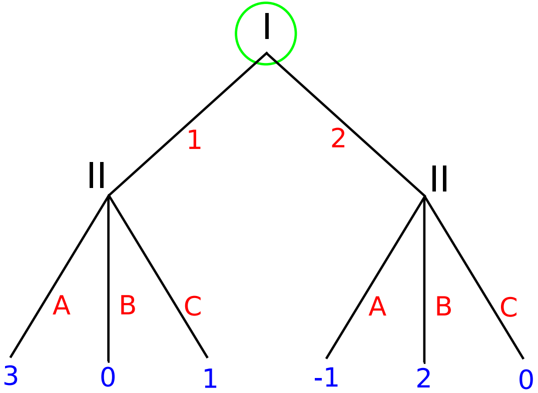
$$A((b_x, b_y), c_z) = \begin{cases} 1 & z < x \leq y \\ \frac{1}{4}(1) + \frac{3}{4}(-1 - y) & y < z \leq x \\ \frac{1}{4}(1 + x) + \frac{3}{4}(-1 - y) & y < x < z \end{cases}$$

Converting Strategic Form to Extensive Form

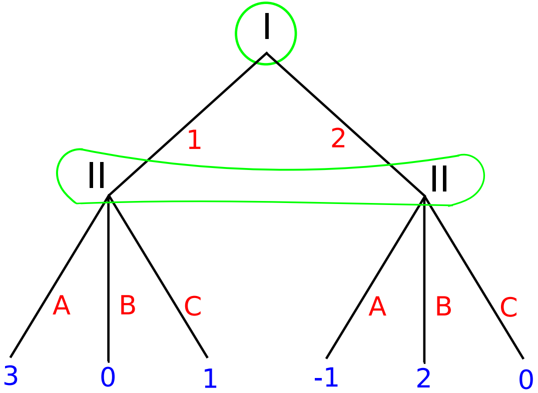
- Last time we discussed going from Extensive Form to Strategic Form. Can we go the other way?
- We can, though we will have lost information about the different "turns" of the game, or about any chance moves.
- **Example.** Draw the following game in Extensive Form.

$$\begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 3 & 0 & 1 \\ -1 & 2 & 0 \end{pmatrix} \end{array}$$

Converting Strategic Form to Extensive Form



Converting Strategic Form to Extensive Form



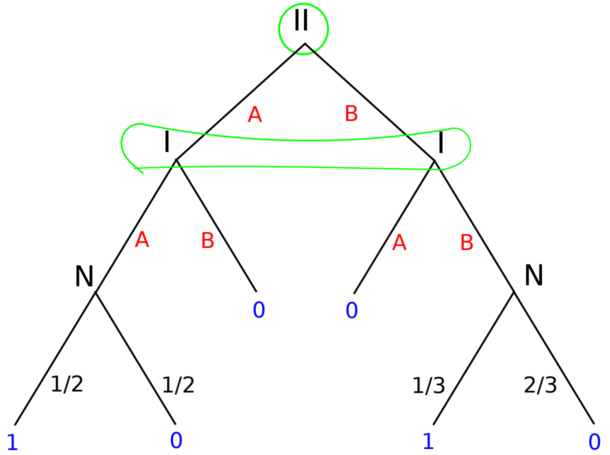
Example

Question

Player II chooses one of two rooms in which to hide a dollar. Player I, not knowing which room Player II chose, can pick one room to search for the dollar. If the dollar is in the first room, and Player I searches this room, the probability that they find the dollar is $1/2$. The second room is messier, so if the dollar is in this room and Player I searches for it, the probability of finding it is $1/3$.

Write this game in Extensive Form and then convert it to Strategic Form and back.

Dollar Game: Extensive Form



Dollar Game: Strategic Form

To convert this to strategic form, we need to know the pure strategies for each player. There are two for each player: either room A or room B.

If they pick different rooms, the payoff is zero. If they both pick room A, the expected payoff is

$$\text{Payoff}(A, A) = \frac{1}{2}(1) + \frac{1}{2}(0) = \frac{1}{2}$$

Similarly for room B. The payoff matrix is:

$$\begin{array}{cc} & \begin{array}{cc} A & A \end{array} \\ \begin{array}{c} A \\ A \end{array} & \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \end{array}$$

Dollar Game: Back to Extensive Form