Math 152: Applicable Mathematics and Computing

May 12, 2017

Announcements

- Midterm 2 is next Friday. Questions like homework questions, plus definitions.
- A list of definitions will be published online over the weekend.
- Homework 5 is due next week. Exam based on Homeworks 3, 4, 5.
- Exam covers Part II, excluding: 4.4-4.6 and 5.8. For *Invariance*, only the material covered in class and in homeworks.
- Don't forget a blue book!

Course Overview

- Part I: Combinatorial Games (Nim, Takeaway Games, Sprague-Grundy) ✓
- Part II: Two Person Zero-Sum Games (Matrix Games, Extensive and Strategic Forms) ✓
- Part III: Two Person General-Sum Games
- Part IV: Many Person Games

General-Sum Games Overview

- Two Player Games
  - General-Sum
    - Noncooperative
      - Find strategies that are good for everyone without making binding agreements.
    - Cooperative
      - No payments
        - Bargaining: threats and counterthreats
      - Side-payments
        - One player bribes the other.
  - Zero-Sum
    - Everyone for themselves

Motivating Example

Game (Prisoner's Dilemma)

Two people are arrested after a bank robbery. They are interrogated by police separately. Each person is confronted with two choices: cooperate with their partner and admit nothing, or defect from their partner and provide evidence against them.

- If both choose to cooperate, they both serve minimal prison time, represented as a payoff of 3 to both players
- If both defect, they both serve longer sentences, represented as a payoff of 1 to both players
- If one defects and one cooperates, the defector gets immunity (payoff of 4) and the cooperator gets a very long sentence (payoff of 0).

Prisoner's Dilemma: Good Strategy?

- This is **not a zero-sum game**. It is a general-sum game, since the sum of Player I and Player II's payoffs are not zero.
- What is a good strategy here?
- It depends what we mean by "good" strategy.
- In one sense, both players cooperating is good, since the total payoff is largest in this case ($3 + 3 = 6$).
- But if we are one of these prisoners, would we choose to cooperate?

Prisoner's Dilemma: Good Strategy?

- Let's consider from Player I's perspective.
- If Player II chooses to cooperate, then our possible payoffs are 3 if we cooperate and 4 if we defect.
- If Player II chooses to defect, then our possible payoffs are 0 if we cooperate and 1 if we defect.
- In **either** situation, it is better if we defect.
- This is a little depressing, encouraging a rather cynical world view. Does this simplified situation miss something important from real-world (criminal) life?

Prisoner's Dilemma: Good Strategy?

- In reality, defection comes at a cost: our *reputation* is burned, we are less trustworthy.
- We can model this by playing the game repeatedly. If they always cooperate, both players are better off.
- Interestingly, if both players know that the game will be played **exactly** N times for some N, then they still have a reason to defect:
  1. In the last game, both players have no reason to cooperate, so will defect.
  2. In the second last game, both players know that in the next turn their opponent will defect. So they defect this turn too.
  3. ...

Aside: I know that you know that I know that you know ...

Question (Island Eye Color Riddle)

The 200 natives of an island have blue eyes or green eyes, 100 people with each color. By a local custom, no one is allowed to know the color of their own eyes, and if they somehow find out they must leave the island that night. The islanders have excellent logical deduction skills, due to the island's free university education.

An anthropologist is visiting the island, and on their last day, in front of all of the islanders, they mention that someone has blue eyes (not a specific person, just that there is at least one person with blue eyes).

Why this was a bad thing for the anthropologist to say?

Aside: I know that you know that I know that you know ...

Question (Island Eye Color Riddle)

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Why this was a bad thing for the anthropologist to say?

(This will **not** be on an exam.)

Prisoner's Dilemma: Strategic Form

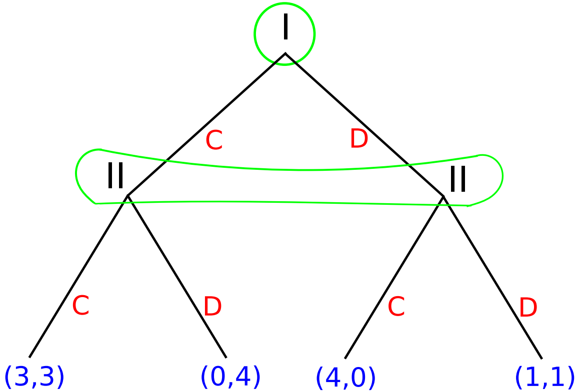
- Like with zero-sum games, we can represent the payoffs in *strategic form*. However, for each pair of strategies we need to give two numbers instead of one.
- The strategic form for the Prisoner's Dilemma is

$$\begin{array}{cc} & \begin{array}{cc} \quad C & \quad\; D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{pmatrix} (3,3) & (0,4) \\ (4,0) & (1,1) \end{pmatrix} \end{array}$$

- The payoff $(0, 4)$ means that Player I receives 0 units and Player II receives 4 units. Note that now **both** players want the payoff to be a positive number.
- We call this a bimatrix.

Prisoner's Dilemma: Strategic Form

Similarly, we also can write general-sum games in Extensive Form.



Safety Levels

- A bimatrix can be broken into two matrices, one for each player. Let us call them A and B for Player I and II respectively.
- **Def.** The safety level of Player I is the maximum amount Player I can guarantee winning, no matter what their opponent does:

$$v_I = \max_p \min_j \sum_{i=1}^{m} p_i a_{ij} = Val(A)$$

- **Def.** The safety level of Player II is the maximum amount Player II can guarantee winning, no matter what their opponent does:

$$v_{II} = \max_q \min_i \sum_{j=1}^{n} a_{ij} q_j = Val(B^T)$$

- To find the safety levels, we just find the values of A and B^T.

Safety Levels Example

Question

For the matrix game

$$G = \begin{pmatrix} (2,0) & (1,3) \\ (0,1) & (3,2) \end{pmatrix}$$

find the safety levels for both players.

Safety Levels Example

In this case, the two player's matrices are:

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 3 \\ 1 & 2 \end{pmatrix}$$

First we find the value of A. There is no saddle point, so we look for an equalizing strategy

$$\begin{aligned} 2p &= V \\ p + 3(1-p) &= V \end{aligned}$$

Solving this we get $p = 3/4$, $V = 3/2$. So $v_I = 3/2$.

Safety Levels Example

In this case, the two player's matrices are:

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 3 \\ 1 & 2 \end{pmatrix}$$

Next we find the value of B^T.

$$B^T = \begin{pmatrix} 0 & 1 \\ 3 & 2 \end{pmatrix}$$

2 is a saddle point, so $v_{II} = 2$.

Safety Levels Example

$$A = \begin{pmatrix} (2,0) & (1,3) \\ (0,1) & (3,2) \end{pmatrix}$$

- We have seen that in this game, Player I has a strategy that guarantees a payoff of at least $3/2$, and Player II can guarantee a payoff of at least 2.
- Notice that each player devised these strategies by **ignoring** their opponent's possible moves.
- We can do better. For example, Player II will choose column 2 by domination. Player I knows this.
- With this knowledge, Player I should pick row 2, for a payoff of 3.

Safety Levels Example 2

Question

Compute the safety levels for the Prisoner's Dilemma.

$$\begin{array}{cc} & \begin{array}{cc} C & D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{pmatrix} (3,3) & (0,4) \\ (4,0) & (1,1) \end{pmatrix} \end{array}$$