Math 152: Applicable Mathematics and Computing

April 6, 2017

Announcements

- First homework is posted on the website, due on Wednesday the 12th at 4PM in APM basement dropboxes.
- Office hours: An-Vy Hoang on Mondays and Wednesdays from 11AM-1PM in APM 6436. Josh Tobin on Mondays at 9AM and 1PM-3PM, and Wednesdays 1PM-2PM. Other office hours to be announced.

P- and N-positions: Example

Question (Subtraction game)

Consider the subtraction game where on each turn a player may remove 1, 3 or 4 coins, and the last player to remove a coin wins. Determine which player will win if there are 15 coins.

What if there are 1000 coins?

Introducing Nim

Game (Nim)

In the game of *Nim*, there are is collection of piles of chips on a table. On each player's turn, they can remove any number of chips from **one** of the piles. Each player must remove at least one chip on their turn. The last player to remove a chip is the winner.

This is the most important impartial game we will study. In fact, we will see that surprisingly *every* impartial game is really Nim in disguise.

Nim: Preliminary Questions

We want to decide which positions are N-positions and which are P-positions). It's **not obvious** for this game!

- We will represent positions in this game as a list of the sizes of each pile: e.g. $(1, 4, 2)$ represents three piles, with 1, 4 and 2 chips.
- **What are the terminal positions?** A position where each pile is empty. E.g. $(0, 0, 0)$ in the game where there were three piles.
- **If there is just one pile, which positions are N/P-positions?** (0) is a P-position, and all other positions are N-positions, since the next player can simply remove all the coins in the pile and win.
- **Who wins from the position $(1, 1)$?** It is a P-position: the first player (**N**) must remove one coin, then the second player (**P**) takes the last coin and wins.

Nim: Two Piles

Conjecture: the P-positions are the positions (x, x). Let's prove this.

Let $\mathcal{P}$ be the set of positions (x, x), and $\mathcal{N}$ be all other positions. Need to show:

1. Terminal positions are in $\mathcal{P}$.
2. From every position in $\mathcal{N}$, there is a move to a position in $\mathcal{P}$.
3. From every position in $\mathcal{P}$, every move is goes to a position in $\mathcal{N}$.

Nim: Two Piles

(Remember: $\mathcal{P} = \{(x, x)\}, \mathcal{N} = \{(x, y) : x \neq y\}$)

1. **Terminal positions are in $\mathcal{P}$:** The terminal position is $(0, 0)$, which is in $\mathcal{P}$.
2. **From any position in $\mathcal{N}$, there is a move to a position in $\mathcal{P}$:** Take $(x, y) \in \mathcal{N}$, so $x \neq y$. We can move to one of (x, x) or (y, y), depending on whether $x < y$ or $y > x$. These are in $\mathcal{P}$.
3. **From any position in $\mathcal{P}$, every move is goes to a position in $\mathcal{N}$:** Take $(x, x) \in \mathcal{P}$. Every move changes the size of exactly one pile, so results in a position (x, y), with $x \neq y$. This is in $\mathcal{N}$.

Writing numbers in base-2

- Before solving Nim for more than two piles, we need to know how to write numbers in base-2 (i.e. in "binary").
- Every non-negative integer can be written uniquely as the sum of unique powers of 2: eg. $19 = 2^4 + 2^1 + 2^0$.
- We can write this as: $19 = 1 \cdot 2^4 + 0 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0$
- For simplicity, we will just record the 1's and 0's: $19 = (10011)_2$.
- If you have never seen this before, it is a good idea to get comfortable with it, since we will be doing this frequently.

Nim-sum: definition

Def. The nim-sum of two numbers in base-2 is defined by

$$(x_m x_{m-1} \cdots x_0) \oplus (y_m y_{m-1} \cdots y_0) = (z_m z_{m-1} \cdots z_0)$$

where $z_i = 1$ if $x_i + y_i = 1$, and $z_i = 0$ if $x_i + y_i = 0$ or $x_i + y_i = 2$.

For example:

- $(10110)_2 \oplus (11101)_2 = (1011)_2$
- $(1110)_2 \oplus (10)_2 = (1100)_2$

Nim-sum: properties

Here are some useful properties of the nim-sum operation:

- $x \oplus y = y \oplus x$ *(commutativity)*
- $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ *(associativity)*
- $x \oplus x = 0$ *(self-cancelling)*
- $x \oplus 0 = 0 \oplus x = x$ *(identity)*
- $x \oplus y = x \oplus z \Rightarrow y = z$ *(cancellation)*

Nim and nim-sim

The connection between nim and nim-sum is given by the following theorem:

Theorem (Bouton, 1902)

A position $(x_1, x_2, \cdots, x_k)$ in a game of Nim is a P-position if and only if

$$x_1 \oplus x_2 \oplus \cdots \oplus x_k = 0$$