Math 152: Applicable Mathematics and Computing

May 17, 2017

Existence of a Strategic Equilibrium

Theorem (Nash, 1950)

Every bimatrix game has at least one mixed strategic equilibrium.

- This was proved by John Nash in 1950, whose life was depicted in the film *A Beautiful Mind*.
- For this reason, strategic equilibria are also called Nash equilibria.
- More generally, he showed that for any finite number of players, each with a finite number of pure strategies, there is at least one (mixed) strategic equilibrium.

Cournot Duopoly

Game

Company A and B produce identical bottles of water. The cost of producing one bottle is c, and the price that consumers are willing to pay is $(a - Q)^+$, where a is a fixed constant and Q is the total number of bottles produced by each firm. Both companies choose the amount they produce simultaneously, and all bottles are then sold at the value given above.

How much should each firm produce?

This question was studied by Antoine Cournot in 1838.

(The notation x^+ means x when $x \geq 0$, and 0 if $x < 0$).

Cournot Duopoly

Game

Company A and B produce bottled water. The cost of producing one bottle is c, and the price that consumers are willing to pay is $(a - Q)^+$, where a is a fixed constant and Q is the total number of bottles produced by each firm.

Let q_1 be the amount produced by Company A, and q_2 the amount produced by Company B. Hence $Q = q_1 + q_2$.
The payoff for Company A is then

$$u_1(q_1, q_2) = q_1(a - q_1 - q_2)^+ - cq_1$$

and for Company B:

$$u_2(q_1, q_2) = q_2(a - q_1 - q_2)^+ - cq_2$$

Cournot Duopoly: Monopoly Situation

- As a first step, let us analyze this game where there is just one company.
- The payoff is

$$u_1(q_1, 0) = q_1(a - q_1)^+ - cq_1$$

- To maximize this, we set the derivative equal to zero:

$$\frac{\partial u_1}{\partial q_1}(q_1, 0) = \begin{cases} a - 2q_1 - c, & a - q_1 \geq 0 \\ -c, & a - q_1 < 0 \end{cases}$$

- This partial derivative is equal to zero only when

$$q_1 = \frac{a - c}{2}$$

- In this case, the payoff is

$$u_1\left(\frac{a-c}{2}, 0\right) = \frac{a-c}{2}\frac{a+c}{2} - \frac{ac - c^2}{2} = \frac{(a-c)^2}{4}$$

Cournot Duopoly: Monopoly Situation

In summary, if there is just one firm:

- The firm will produce $(a-c)/2$.
- The profit is $(a-c)^2/4$.
- The cost, per bottle, to the consumer is

$$(a-Q)^+ = \frac{a+c}{2}$$

Cournot Duopoly: Company A

- Now we return to the situation with two companies.
- For a fixed q_2, we can find the values of q_1 where Company A has no incentive to change q_1. Again, we find when the derivative is zero:

$$\frac{\partial u_1}{\partial q_1}(q_1, q_2) = \begin{cases} a - 2q_1 - q_2 - c, & a - q_1 - q_2 \geq 0 \\ -c, & a - q_1 - q_2 < 0 \end{cases}$$

- Solving, we get

$$q_1 = \frac{a - c - q_2}{2}$$

- The interpretation is: for a fixed value of q_2, Company A's best response is $q_1 = (a - c - q_2)/2$.

Cournot Duopoly: Company B

- We have seen that for a fixed q_2, Company A's best response is $q_1 = (a - c - q_2)/2$.
- By symmetry, for a fixed q_2, Company B's best response is $q_2 = (a - c - q_1)/2$.
- If we solve these simultaneously, we find a pair of strategies that are best responses to each other.
- Since neither player has incentive to switch, this is a pure strategic equilibrium.
- Solving, we find

$$q_1 = q_2 = \frac{a - c}{3}$$

Cournot Duopoly

- So we have found a pure strategic equilibrium with

$$q_1 = q_2 = \frac{a-c}{3}$$

- The profits for each company are given by

$$\frac{a-c}{3} \cdot \left(a - \frac{2(a-c)}{3} - c \right) = \frac{(a-c)^2}{9}$$

- The cost, per bottle, to the consumer is given by

$$\left(a - \frac{2(a-c)}{3} \right) = \frac{a+2c}{3}$$

Duopoly vs Monopoly Comparison

	Amount Produced	Price	Company A	Company B
Monopoly	$\frac{a-c}{2}$	$\frac{a+c}{2}$	$\frac{(a-c)^2}{4}$	0
Duopoly	$\frac{2(a-c)}{3}$	$\frac{a+2c}{3}$	$\frac{(a-c)^2}{9}$	$\frac{(a-c)^2}{9}$

Cournot Duopoly: Other Equilibria?

- We have found a pure strategic equilibrium for the Cournot Duopoly setup.
- Could there be another equilibrium?
- **Note:** There are uncountably many pure strategies (any nonnegative real). So a mixed strategy here is *any probability distribution on the nonnegative reals*. This is a huge space!
- Luckily, a minor miracle occurs: we can iteratively remove dominated strategies, and all that will be left is the equilibrium we found above.

Cournot Duopoly: Removing Dominated Strategies

- Recall that
$$u_1(q_1, q_2) = q_1(a - q_1 - q_1)^+ - cq_1$$
- A quick computation gives
$$\frac{\partial u_1}{\partial q_1}(q_1, q_2) < 0$$
iff $q_1 > (a - c - q_2)/2$.
- Notice that if $q_1 > (a - c)/2$, then this inequality is satisfied for all q_2. It follows that $q_1 = (a - c)/2$ dominates any $q_1 > (a - c)/2$.
- So we can restrict our attention to $q_1 \leq (a - c)/2$.
- By symmetry, we can also assume that $q_2 \leq (a - c)/2$.

Cournot Duopoly: Removing Dominated Strategies

- By domination, we have found that $q_1 \leq (a-c)/2$ and $q_2 \leq (a-c)/2$.
- A quick computation gives

$$\frac{\partial u_1}{\partial q_1}(q_1, q_2) > 0$$

  iff $q_1 < (a-c-q_2)/2$.
- Since $q_2 \leq (a-c)/2$, we have that

$$\frac{a-c}{4} \leq \frac{a-c-q_2}{2}$$

- In particular, if $q_1 < (a-c)/4$, we get that u_1 is increasing, for all q_2.
- So by domination we can assume $q_1 \geq (a-c)/4$

Cournot Duopoly: Removing Dominated Strategies

In summary:

- Using domination, we removed all strategies where q_1, q_2 were larger than $(a-c)/2$.
- With these removed, by domination we could remove all strategies where q_1, q_2 were smaller than $(a-c)/4$.
- With these removed, we can remove even more strategies by domination...
- If we continue this process, one can verify that all strategies will be dominated except $q_1 = q_2 = (a-c)/3$.

Cournot Duopoly: General Pricing

Question

Consider a modification of the Cournot setup, where the two companies have two different production costs, c_1 and c_2, find all pure strategic equilibria.