Math 152: Applicable Mathematics and Computing

May 22, 2017

Bertrand Duopoly: Undifferentiated Products

Game (Bertrand)

Firm I and Firm II produce identical products. Each firm simultaneously decides on the price per unit of their product. Whichever firm sets a smaller price captures the entire market. The total demand is $Q = (a - P)^+$, where P is the smaller of the two prices and a is a constant. If both firms set equal prices, they split the market equally. The cost of producing one unit is c.

This model was studied by Joseph Bertrand in 1883, as an alternative approach to Cournot.

(**Recall**: In the Cournot model, firms set production volume and price is derived from this. In Bertrand, firms set price and production is derived from this.)

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The payoff function for Firm I is:

$$u_1(p_1, p_2) = \begin{cases} (p_1 - c)(a - p_1)^+ & p_1 < p_2 \\ (p_1 - c)(a - p_1)^+/2 & p_1 = p_2 \\ 0 & p_1 > p_2 \end{cases}$$

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In Cournot, the relationship between price and production volume was given by $P = (a - Q)^+$. Whenever profits are positive, this is the same as $Q = (a - P)^+$.

In particular, in a monopoly situation, the analysis for the Bertrand and Cournot models are exactly the same.

Bertrand Duopoly: Undifferentiated Products

- What are the strategic equilibria in this model?
- Notice that if $p_1 > c$, then Firm II's best response is to set $p_2 = p_1 - \varepsilon$, thereby capturing the entire market.
- If either $p_1 < c$ or $p_2 < c$, then the lower-price firm will increase their profits by setting the price to c. So in an equilibrium, $p_1 \geq c$ and $p_2 \geq c$.
- If $p_1 > c$, then $p_2 = p_1 - \varepsilon > c$. But then p_1 should be $p_2 - \varepsilon$. This contradiction implies $p_1 \leq c$ and $p_2 \leq c$.
- Hence $p_1 = p_2 = c$.

Bertrand: Duopoly vs Monopoly Comparison

	Amount Produced	Price	Payoff I	Payoff II
Monopoly	$\frac{a-c}{2}$	$\frac{a+c}{2}$	$\frac{(a-c)^2}{4}$	0
Duopoly	$a-c$	c	0	0

- The result here is counterintuitive: in a dupoly, neither firm will make a profit?
- One alternative approach is due to Francis Edgeworth (1889): firms may be unwilling or unable to meet all demand if the price is too low.
- In this case, if $p_1 = p_2 = c$, then both firms have an incentive to increase price slightly, as they will produce sell more goods.

Bertrand Duopoly: Differentiated Products

Game (Bertrand, Differentiated Products)

Firm I and Firm II produce different, but similar, products. Each firm simultaneously decides on the price per unit of their product, denoted p_1 and p_2 respectively. The demands for each product are given by

$$\begin{aligned} q_1(p_1, p_2) &= (a - p_1 + bp_2)^+ \\ q_2(p_1, p_2) &= (a - p_2 + bp_1)^+ \end{aligned}$$

where a, b are positive constants with $b \leq 1$. The cost of producing one unit is c.

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where a, b are positive constants with $b \leq 1$. The cost of producing one unit is c.

The payoff functions in this case are given by:

$$\begin{aligned} u_1(p_1, p_2) &= (a - p_1 + bp_2)^+(p_1 - c) \\ u_2(p_1, p_2) &= (a - p_2 + bp_1)^+(p_2 - c) \end{aligned}$$

Bertrand Duopoly: Differentiated Products

- We want to find the strategic equilibria of the game with payoff functions

$$\begin{aligned} u_1(p_1, p_2) &= (a - p_1 + bp_2)^+(p_1 - c) \\ u_2(p_1, p_2) &= (a - p_2 + bp_1)^+(p_2 - c) \end{aligned}$$

- We fix p_2, and then maximize u_1 as p_1 varies.

$$\frac{\partial u_1}{\partial p_1}(p_1, p_2) = \begin{cases} 0 & a - p_1 + bp_2 < 0 \\ a - 2p_1 + bp_2 + c & a - p_1 + bp_2 \geq 0 \end{cases}$$

- So the best response strategy for Firm I is

$$p_1 = \frac{a + bp_2 + c}{2}$$

Bertrand Duopoly: Differentiated Products

- We have deduced that

$$p_1 = \frac{a + bp_2 + c}{2}$$

- By symmetry, we have

$$p_2 = \frac{a + bp_1 + c}{2}$$

- Solving this, we get

$$p_1 = p_2 = \frac{a + c}{2 - b}$$

- **Note:** If $b = 0$, this means neither product is a replacement for the other. This situation degenerates into two monopolies, and we have that $p_1 = (a + c)/2$ which agrees with what we found before.

Bertrand Duopoly: Differentiated Products

- We have found a strategic equilibrium with

$$p_1 = p_2 = \frac{a + c}{2 - b}$$

- In this case, the demand for both products is gien by

$$q_1 = q_2 = \frac{a - c + bc}{2 - b}$$

- The profit for each company is given by

$$u_1(p_1, p_2) = u_2(p_1, p_2) = \left(\frac{a - c + bc}{2 - b}\right)^2$$

Stackelberg Duopoly

Game (Heinrich von Stackelberg, 1934)

Firm I and Firm II produce identical products. Firm I will decide its production amount q_1 for some fiscal period, and announce this publically. Firm II will then decide its production amount q_2. The price of a unit of this product is then given by $(a - q_1 - q_2)^+$, for some constant a. The cost of producing one unit of the product is c.

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The payoffs here are the same as under Cournot:

$$\begin{aligned} u_1(q_1, q_2) &= q_1(a - q_1 - q_2)^+ - cq_1 \\ u_2(q_1, q_2) &= q_2(a - q_1 - q_2)^+ - cq_2 \end{aligned}$$

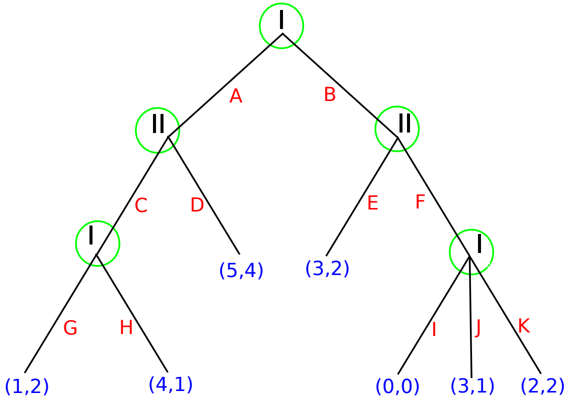
The difference with Cournot is that now Firm II knows Firm I's strategy.

Games of Perfect Information

- **Recall:** A game of perfect information is any game where every information set consists of exactly one vertex.
- In a finite game of perfect information, we can solve it by iteratively removing dominated strategies, starting from the bottom (so this is an instance of backwards induction).
- The Stackelberg model is a game of perfect information.

Games of Perfect Information: Example

For example, we solve the game of perfect information below.



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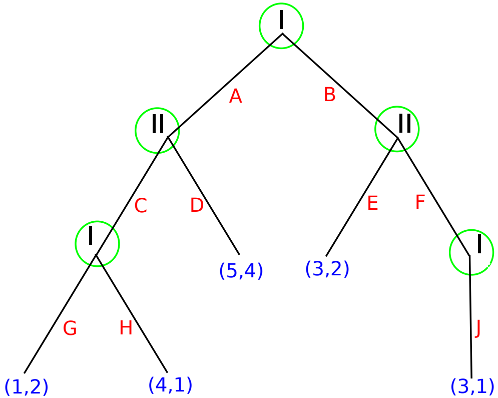

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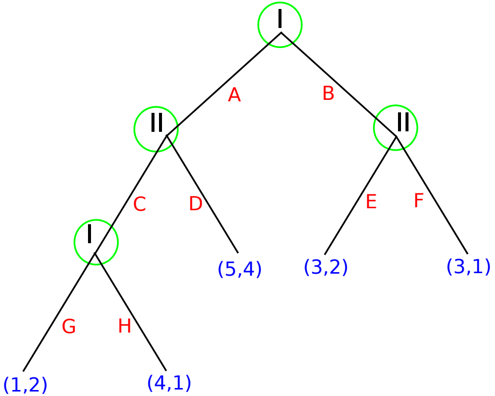

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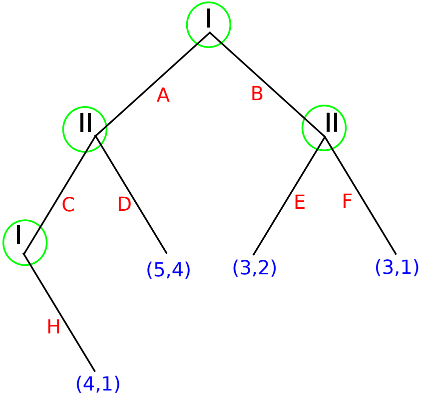

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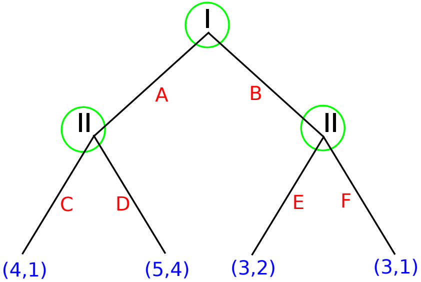

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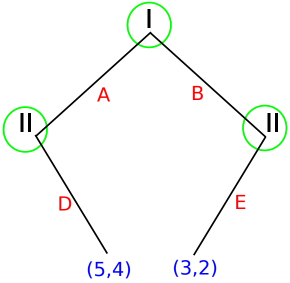

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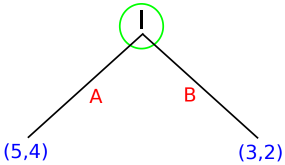

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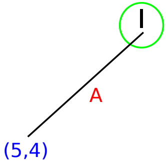

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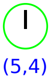

- Let us return to Stackelberg.
- We have a game of perfect information, where Firm I moves first with strategy q_1 and then Firm II chooses their strategy q_2. The payoffs are:

$$\begin{aligned} u_1(q_1, q_2) &= q_1(a - q_1 - q_2)^+ - cq_1 \\ u_2(q_1, q_2) &= q_2(a - q_1 - q_2)^+ - cq_2 \end{aligned}$$

- Firm II will choose a best response to q_1.
- By calculus, just like for Cournot, this best response is

$$q_2 = \frac{a - c - q_1}{2}$$

- Firm I knows this is what Firm II will do. So Firm I knows their payoff is

$$u_1(q_1) = q_1 \left(a - q_1 - \frac{a - c - q_1}{2} \right)^+ - cq_1$$

- Firm II knows their payoff is

$$u_1(q_1) = q_1 \left(a - q_1 - \frac{a - c - q_1}{2} \right)^+ - cq_1 = q_1 \left(\frac{a - q_1 + c}{2} \right)^+ - cq_1$$

- So they will maximize this over q_1. We have

$$\frac{\partial u_1}{\partial q_1} = \begin{cases} \frac{a - 2q_1 - c}{2} & a - q_1 + c \geq 0 \\ 0 & a - q_1 + c < 0 \end{cases}$$

- The max occurs when $q_1 = (a - c)/2$.
- This gives $q_2 = (a - c)/4$.

Bertrand: Duopoly vs Monopoly Comparison

	Amount Produced	Price	Payoff I	Payoff II
Cournot	$\frac{2(a-c)}{3}$	$\frac{a+2c}{3}$	$\frac{(a-c)^2}{9}$	$\frac{(a-c)^2}{9}$
Stackelberg	$\frac{3(a-c)}{4}$	$\frac{a+3c}{4}$	$\frac{(a-c)^2}{8}$	$\frac{(a-c)^2}{16}$

So under Stackelberg, Firm I does better than under Cournot. Additionally, the consumer is better off.

Remark: It seems that Firm II has an advantage in Stackelberg, since they know Firm I's strategy. But Firm I *knows* that Firm II knows Firm I's strategy. This gives Firm I control.

Hedge Fund Interview Question

Game (Quant Interview Question)

Two players each draw a uniform random number from the interval $[0, 1]$. If a player does not like their number, they can redraw a new number, but only once. Neither player knows the other's number, or whether their opponent has redrawn. The player with the higher number wins 1.

What are the optimal strategies for each player?

The answer may be a little counterintuitive. (This won't be on an exam).