Math 152: Applicable Mathematics and Computing

May 24, 2017

Announcements

- Midterm scores will be posted later today.
- The average was 85%.
- Keep in mind: there will be no definitions on the final (and people did very well on the definitions)..
- Homework 6 will be posted this afternoon.

Entry Deterrence

Game

Consider a monopoly, where the price demand relationship is given by

$$P(Q) = \begin{cases} 17 - Q & 0 \leq Q \leq 17 \\ 0 & \text{otherwise} \end{cases}$$

and where the cost of producing q_1 units is $q_1 + 9$.

What is the best choice of q_1?

(**Remark**. Note that the production cost has a startup cost here, unlike our previous models.)

Entry Deterrence

- In this case, the payoff function is

$$u_1(q_1) = q_1(17 - q_1) - (q_1 + 9)$$

- The derivative is
$$\frac{du_1}{q_1} = -2q_1 + 16,$$
so the maximum will be obtained at $q_1 = 8$.
- So the payoff is $u_1 = 8 \cdot 9 - 17 = 55$.

Entry Deterrence

- After this firm has decided to set $q_1 = 8$, another firm considers entering the market.
- They calculate that their payoff will be

$$u_2(q_2) = q_2(17 - 8 - q_2) - (q_2 + 9)$$

- They optimize this,

$$\frac{du_2}{q_2} = -2q_2 + 8$$

- So the max occurs when $q_2 = 4$, yielding a profit of $u_2(4) = 7$.
- But now the first firm's profits *decrease*. Firm I does not like this. If they could go back in time, can they prevent this?

Entry Deterrence

- Firm I is worried that a competitor will choose to produce q_2 units. The payoff for Firm II in this case is

$$u_2(q_1, q_2) = q_2(17 - q_1 - q_2) - (q_2 + 9)$$

- Firm I would like to choose q_1 in such a way that Firm II has no incentive to enter the market.
- For a constant q_1, let's find Firm II's response strategy q_2.
- Maximizing for q_2,

$$\frac{\partial u_2}{\partial q_2} = 17 - q_1 - 2q_2 - 1$$

So $q_2 = 8 - q_1/2$.

Entry Deterrence

- The payoff function is

$$u_2(q_1, q_2) = q_2(17 - q_1 - q_2) - (q_2 + 9)$$

- Firm II's strategy will be $q_2 = 8 - q_1/2$, so

$$\begin{aligned} u_2(q_1, q_2) &= \left(8 - \frac{q_1}{2}\right)\left(17 - q_1 - 8 + \frac{q_1}{2}\right) - \left(8 - \frac{q_1}{2} + 9\right) \\ &= \frac{q_1^2}{4} - 8q_1 + 55 \end{aligned}$$

- If we make this function 0, then Firm II will not enter the market.
- So $\frac{q_1^2}{4} - 8q_1 + 55 = 0 \Rightarrow (q_1 - 10)(q_1 - 22) = 0$
- If $q_1 = 10$, then Firm I's profits are $u_1 = 51$ (compare this to their initial profit of 55).

Entry Deterrence

- How does the situation differ if both firms choose production simultaneously (ie. Cournot)?
- Payoffs in this case are:

$$\begin{aligned} u_1(q_1, q_2) &= q_1(17 - q_1 - q_2) - (q_1 + 9) \\ u_2(q_1, q_2) &= q_2(17 - q_1 - q_2) - (q_2 + 9) \end{aligned}$$

- Setting derivatives to zero, we get

$$-2q_1 + 16 - q_2 = 0 = -2q_2 + 16 - q_1$$

- So $q_1 = q_2 = 16/3$, yielding profits of $19\frac{4}{9}$ (compare this to 55 and 51).

Cooperative Games

- The general-sum games we have been discussing so far have been non-cooperative.
- As we have seen, a strategic equilibrium is self-enforcing: players have no incentive to switch from it, and so are inclined to choose these strategies without needing a binding agreement.
- However, if we allow binding agreements, players can do better.
- For example, the Prisoner's dilemma:

$$\begin{array}{cc} & \begin{array}{cc} C & D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{pmatrix} (3,3) & (0,4) \\ (4,0) & (1,1) \end{pmatrix} \end{array}$$

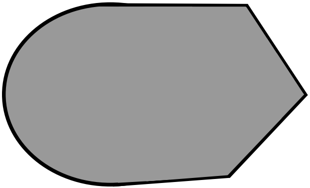
- If possible, the two prisoners will make a binding agreement to both Cooperate.

Cooperative Games: Transferable Utility?

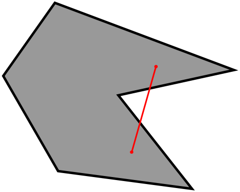
- In the cooperative theory we allow players to make binding agreements.
- We will consider two cases:
  (1) Transferable Utility (TU): players are allowed to make payments to each other when the game ends (called side-payments).
  (2) Nontransferable Utility (NTU): players are not allowed to make side-payments. The only payoff that occurs is from the game itself.

Convex Sets

Def. A subset S of $\mathbb{R}^2$ is a convex set if every straight line joining two points in S is completely contained in S.



Convex



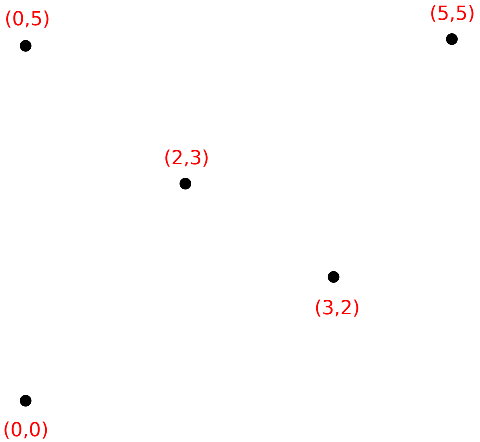
Not Convex

Convex Hull

- **Def.** The convex hull of a set T in $\mathbb{R}^2$ is the smallest convex set S that contains T.
- Given a finite set of points, we can easily find the convex hull, adding one point at a time.
- Finding the convex hull is an important problem in computational geometry, and there are many algorithms (eg. the Graham Scan).

Convex Hull

Example. Find the convex hull of $(0,0),(0,5),(2,3),(3,2),(5,5)$.



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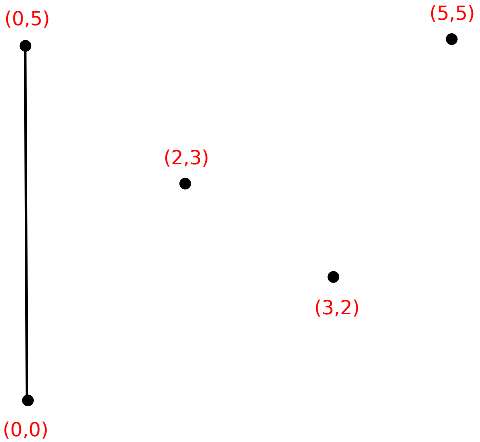

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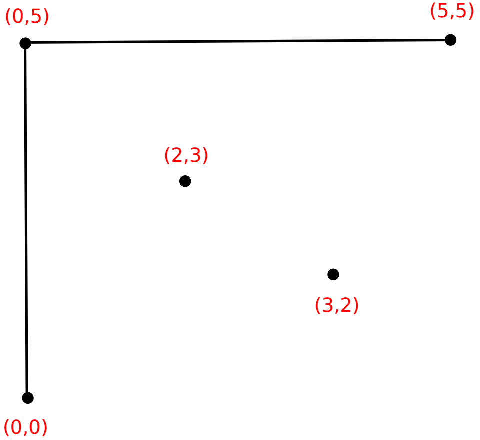
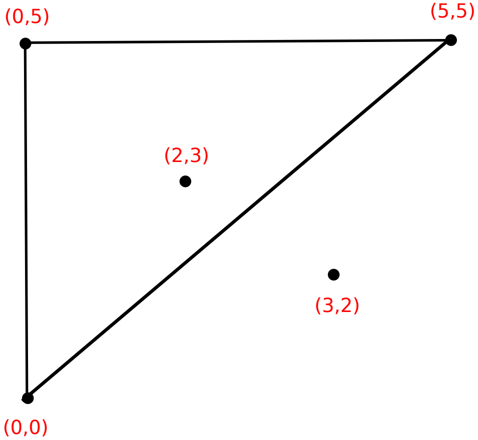


Convex Hull

Example. Find the convex hull of $(0, 0), (0, 5), (2, 3), (3, 2), (5, 5)$.



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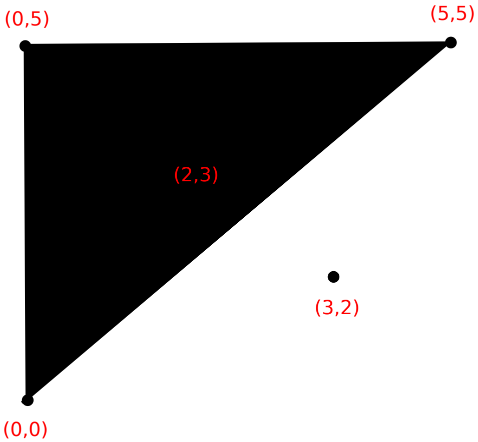

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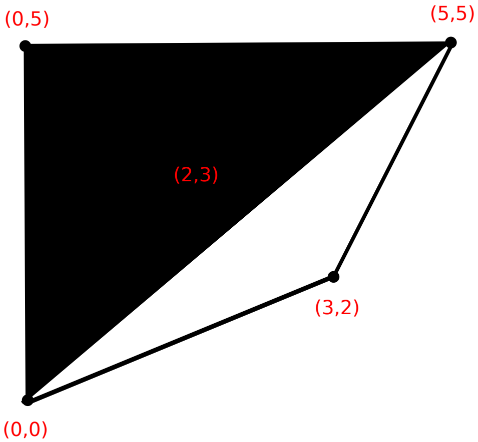

Convex Hull

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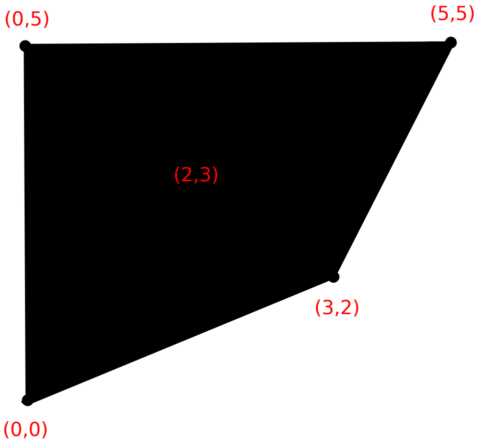

Feasible Sets

- Informally speaking, the *feasible set* of a game is the set of all possible payoffs that might occur as a result of the game (including side-payments, if allowed).
- **Def.** Consider a bimatrix game with payoff matrices A, B, where Player I has m pure strategies and Player II has n pure strategies. The NTU feasible set of this game is the convex hull of the mn points (a_{ij}, b_{ij}), where $1 \leq i \leq m$ and $1 \leq j \leq n$.

Feasible Sets

Question

For the bimatrix game

$$\begin{pmatrix} (4,3) & (0,0) \\ (2,2) & (1,4) \end{pmatrix}$$

find the NTU feasible set.

Feasible Sets

- In a TU game, if (x, y) is a possible payoff in a game, then so is $(x - s, y + s)$ for any constant $s \in \mathbb{R}$. Here s is some sidepayment.
- **Def.** Consider a bimatrix game with payoff matrices A, B. The TU feasible set of this game is the convex hull of the points $(a_{ij} + s, b_{ij} - s)$, where $1 \leq i \leq m$ and $1 \leq j \leq n$ and s is any real number.
- Note that for any point $(x, y) \in \mathbb{R}^2$, the set of points $(x - s, y + s)$ is just the line with slope -1 that goes through (x, y).
- In particular, the TU feasible set is just the NTU feasible set translated along the line with slope -1.

Feasible Sets

Question

For the bimatrix game

$$\begin{pmatrix} (4,3) & (0,0) \\ (2,2) & (1,4) \end{pmatrix}$$

find the TU feasible set.

Pareto Optimality

- **Def.** A payoff (x, y) in the feasible set of a game is Pareto optimal if there is no other point in the feasible set (x', y') with $x' \geq x$ and $y' \geq y$.
- In the diagram of a feasible set, the points which are Pareto optimal will be the upper right boundary.
- When the players come to a binding agreement, they will always choose a point that is Pareto optimal.

Solving TU Games

- The first thing to notice about solving a TU game is that both players will want to choose a point where the sum of the payoffs is maximized.

$$\sigma = \max_i \max_j (a_{ij} + b_{ij})$$

- What remains is to decide how to split σ between the two players.
- Both players will choose a threat strategy that they will follow if negotiations break down. Denote these by $\mathbf{p}$ and $\mathbf{q}$. The payoffs in this case are:

$$(\mathbf{p}^T A\mathbf{q}, \mathbf{p}^T B\mathbf{q}) = (D_1, D_2)$$

- Player I will accept no less than D_1, Player II will accept no less than D_2. These correspond to the payoffs: $(D_1, \sigma - D_1)$ and $(\sigma - D_2, D_2)$.

Solving TU Games

- So Player I will accept no less than $(D_1, \sigma - D_1)$ and Player II will accept no less than $(\sigma - D_2, D_2)$, where $D_1 = \mathbf{p}^T A\mathbf{q}$ and $D_2 = \mathbf{p}^T B\mathbf{q}$.
- A natural compromise is the midpoint:

$$\left(\frac{\sigma + D_1 - D_2}{2}, \frac{\sigma - D_1 + D_2}{2}\right)$$

- So Player I wants to maximize

$$D_1 - D_2 = \mathbf{p}^T(A - B)\mathbf{q}$$

  and Player II wants to minimize it.
- This is equivalent to playing the zero sum game $A - B$.
- If $\delta = \text{Val}(A - B)$, then the TU solution is given by

$$\left(\frac{\sigma + \delta}{2}, \frac{\sigma - \delta}{2}\right)$$

Solving TU Games

Question

Find the TU solution to the game

$$\begin{pmatrix} (0,0) & (6,2) & (-1,2) \\ (4,-1) & (3,6) & (5,5) \end{pmatrix}$$

We know that the solution is given by

$$\left(\frac{\sigma+\delta}{2}, \frac{\sigma-\delta}{2}\right),$$

where

$$\sigma = \max_i \max_j (a_{ij} + b_{ij})$$

and

$$\delta = \mathrm{Val}(A - B)$$