Math 152: Applicable Mathematics and Computing

May 26, 2017

Announcements

- Homework 6 was posted on Wednesday, due next Wednesday.
- Last homework is Homework 7, posted next week (due Week 10).
- Monday is Memorial day: no sections or class.

Solving TU Games

Question

Find the TU solution to the game

$$\begin{pmatrix} (0,0) & (6,2) & (-1,2) \\ (4,-1) & (3,6) & (5,5) \end{pmatrix}$$

We know that the solution is given by

$$\left(\frac{\sigma+\delta}{2}, \frac{\sigma-\delta}{2}\right),$$

where

$$\sigma = \max_i \max_j (a_{ij} + b_{ij})$$

and

$$\delta = \mathrm{Val}(A - B)$$

Solving TU Games

- We have $\sigma = 5 + 5 = 10$.
- The matrix $A - B$ is

$$\begin{pmatrix} 0 & 4 & -3 \\ 5 & -3 & 0 \end{pmatrix}$$

- The third column dominates the first, leaving

$$\begin{pmatrix} 4 & -3 \\ -3 & 0 \end{pmatrix}$$

  The value of this game is $\delta = -9/10$. So the TU solution is $((10 - 9/10)/2, (10 + 9/10)/2) = (4.55, 5.45)$.
- So the sidepayment is $5 - 4.55 = 0.45$ from Player I to Player II.

Solving NTU Games

- John Nash proposed a general model for solving NTU games in 1950 ("The Bargaining Problem", *Econometrica*, Vol. **18**, No. 2, pp. 155–162).
- The inputs are a closed, bounded, convex set S (which represents the feasible payoffs) and a point $(u^*, v^*) \in S$ called the threat point or status quo point.
- The approach is to make some reasonable assumptions about the solution that the players will agree on, and then to show that these assumptions uniquely determine a solution.
- We denote this solution by $f(S, u^*, v^*) = (\overline{u}, \overline{v})$.

Nash Bargaining Axioms

(1) Feasibility: $(\overline{u}, \overline{v}) \in S$.

- This axiom just asserts that the agreed solution is one of the possible payoffs.

(2) Pareto optimality: there is no point $(u, v) \in S$, other than $(\overline{u}, \overline{v})$, that satisfies $u \geq \overline{u}$ and $v \geq \overline{v}$.

- They will not agree on a payoff if they can both achieve more than that payoff (or if one can achieve more, while the other's payoff stays the same).

(3) Symmetry: if S is symmetric about the line $u = v$, and if $u^* = v^*$, then $\overline{u} = \overline{v}$.

- In a symmetric game with equal threats, the agreed upon payoff will be equal for both players.

Nash Bargaining Axioms

(4) Independence of irrelevant alternatives: if T is a closed, convex subset of S, and if $(u^*, v^*) \in T$ and $(\overline{u}, \overline{v}) \in T$, then $f(T, u^*, v^*) = f(S, u^*, v^*) = (\overline{u}, \overline{v})$.

- Removing possible payoffs that are not part of the solution should not change the bargaining process.
- Nash admits that this is the toughest axiom to justify.

(5) Invariance under change of location and scale: let g_1, g_2 be linear transformations, $g_1(t) = \alpha_1 t + \beta_1$, $g_2(t) = \alpha_2 t + \beta_2$, where $\alpha_1 > 0$ and $\alpha_2 > 0$. Then $f(T, g_1(u^*), g_2(v^*)) = (g_1(\overline{u}), g_2(\overline{v}))$, where $T = \{(g_1(u), g_2(v)) : (u, v) \in S\}$

- This just says that if we do a change-of-coordinates then the solution will just be the new coordinates of the old solution.

Nash Bargaining Example

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Let $T = \{(u, v) : u, v \geq 0, u + v \leq 2\}$. If the threat point is $(0, 0)$, what is the NTU solution $f(T, 0, 0)$?

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- Axiom 1 implies that $f(T, 0, 0) \in T$.
- Axiom 2 implies that $f(T, 0, 0)$ lies on the line $u + v = 2$.
- Axiom 3 implies that $f(T, 0, 0) = (u, u)$ for some $u \in \mathbb{R}$.

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- Axiom 3 implies that $f(T, 0, 0) = (u, u)$ for some $u \in \mathbb{R}$.
- So we get $u + u = 2 \Rightarrow f(T, 0, 0) = (1, 1)$.

Nash Bargaining Base Case

Corollary

Let T be any closed, bounded, convex subset of $\mathbb{R}^2$ that contains the points $(0,0),(1,1)$ and lies on or below the line $u+v=2$. Then $f(T,0,0)=(1,1)$, and moreover $(1,1)$ is the *only* point that satisfies the Nash axioms.

Proof.

This just follows from the solution to the previous example, together with Axiom 4. □

Nash Bargaining Solution

Theorem

There is a unique function satisfying the Nash axioms. Moreover, if there is a point $(u, v) \in S$ with $u > u^*$ and $v > v^*$, then $f(S, u^*, v^*)$ is exactly the point which maximizes $(u - u^*)(v - v^*)$ over all $u \geq u^*$, $v \geq v^*$.

Proof.

Let $(\overline{u}, \overline{v})$ be the point that maximies $(u - u^*)(v - v^*)$.
Let g_1 be the linear transformation sending $u^* \mapsto 0$ and $\overline{u} \mapsto 1$, and g_2 be the linear transformation sending $v^* \mapsto 0$ and $\overline{v} \mapsto 1$. Let $T = \{(g_1(u), g_2(v)) : (u, v) \in S\}$. Then:

$$(\overline{u}, \overline{v}) \text{ maximizes } (u - u^*)(v - v^*) \text{ in } S$$

$$\Updownarrow$$

$$(1, 1) \text{ maximizes } uv \text{ in } T$$

Nash Bargaining Solution

Proof continued...

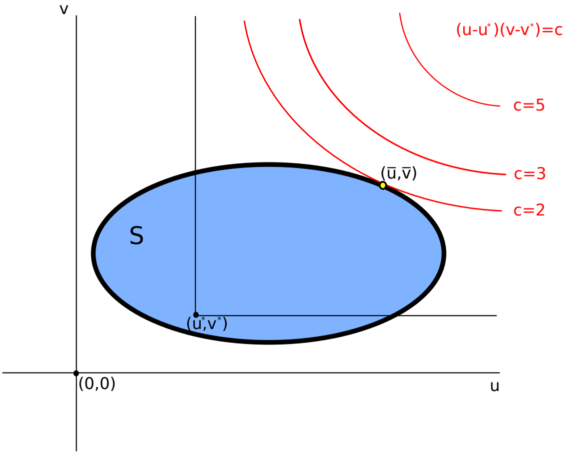
Now since T is convex, this means T is contained below the line $u + v = 2$.

By our corollary, this means $f(T, 0, 0) = (1, 1)$.

Now by axiom 5 that means $f(S, u^*, v^*) = (\overline{u}, \overline{v})$.

It is easy to verify that $(\overline{u}, \overline{v})$ satisfy axioms 1 - 3. So we are done. □

Nash Bargaining: Geometric Interpretation



Note that the slope of the curve at $(\overline{u}, \overline{v})$ is the negative of the slope of the line from (u^*, v^*) to $(\overline{u}, \overline{v})$. *Why?*

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Let S be the triangle with vertices $(0,0),(0,1),(3,0)$. Find $f(S,0,0)$.

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- So the slope of the line from $(0,0)$ to $(\overline{u},\overline{v})$ must be $1/3$.
- The intersection of the line joining $(0,1)$ to $(3,0)$ and the line through $(0,0)$ with slope $1/3$ is the point $(3/2,1/2)$. So this is our NTU solution.

Nash Bargaining Example 2

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- By axiom 2, we know that the solution lies on the part of the curve between the points $(2, 1+\sqrt{2})$ and $(2+2\sqrt{2}, 1)$. On this part of the curve we have

$$y - 1 = \sqrt{2 - \frac{(x-2)^2}{4}}$$

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- We need to maximize $(x-2)(y-1)$ along this arc, and

$$(x-2)(y-1) = (x-2)\sqrt{2 - \frac{(x-2)^2}{4}}$$

- Setting the derivative of this to zero, we get $x = 2 \pm 2$. $x > 2$, so $x = 4$. Then $y = 2$, so $(4, 2)$ is our solution.

Nash Bargaining Example 3

Question

For the bimatrix game

$$\begin{pmatrix} (4,3) & (0,0) \\ (2,2) & (1,4) \end{pmatrix}$$

find the NTU solution if the threat point is $(0, 0)$.

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However if we take the line from $(0,0)$ with slope $1/3$, it does not intersect the Pareto optimal boundary of the triangle at all. (It is below the triangle).

This means that uv is an increasing function along the upper right boundary of the triangle. So the max occurs at the corner $(4,3)$.

λ-Transfer Approach

- An alternative approach to NTU games follows from observing:
  - If the TU solution is in the NTU feasible set, we are done.
  - In an NTU game, we can rescale utility for an individual player.
- Given a bimatrix game (A, B), we scale one player's payoffs and TU solve this new game. If the solution is in the NTU feasible set, we have the NTU solution too.
- Then we take this solution and rescale the payoffs back to the original amount.

λ-Transfer Approach

- Let (A, B) be a given bimatrix game. We scale Player I's payoffs by λ, for some λ to be chosen later. The new game is $(\lambda A, B)$.
- Recall that the TU solution to this game is given by

$$\left(\frac{\sigma(\lambda)+\delta(\lambda)}{2}, \frac{\sigma(\lambda)-\delta(\lambda)}{2}\right)$$

  where $\sigma(\lambda)$ is the largest entry in the matrix $\lambda A + B$ and $\delta(\lambda)$ is the value of $\lambda A - B$.
- We vary λ until this solution is in the NTU feasible set of $(\lambda A, B)$.
- Then the NTU solution to (A, B) is given by

$$\left(\frac{\sigma(\lambda)+\delta(\lambda)}{2\lambda}, \frac{\sigma(\lambda)-\delta(\lambda)}{2}\right)$$

- It was shown by Lloyd Shapley that there is a unique value of λ that will work (1967).