Math 152: Applicable Mathematics and Computing

May 31, 2017

Announcements

- Homework 7 is posted, due Wednesday of Week 10.
- The last three questions are not to be turned in, as they are based on material we will cover on Friday and early next week. (They are included in the homework to indicate that this material may appear on the exam.)
- As indicated in the syllabus, from Part IV we will cover chapters 1 and 3.

Solving NTU Games

Last time, we saw:

- The NTU solution lies on the Pareto optimal curve of the NTU feasible set (Axiom 2).
- The NTU solution $(\overline{u}, \overline{v})$ maximizes $(u - u^*)(v - v^*)$ where (u^*, v^*) was the threat point and $u \geq u^*$, $v \geq v^*$.
- The slope of the curve at $(\overline{u}, \overline{v})$ is the negative of the slope of the line from (u^*, v^*) to $(\overline{u}, \overline{v})$.

From Last Time: Nash Bargaining Example 1

Question

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- We know the solution is on the line joining $(0,1)$ to $(3,0)$. This curve as slope $-1/3$.
- So the slope of the line from $(0,0)$ to $(\overline{u},\overline{v})$ must be $1/3$.
- The intersection of the line joining $(0,1)$ to $(3,0)$ and the line through $(0,0)$ with slope $1/3$ is the point $(3/2,1/2)$. So this is our NTU solution.

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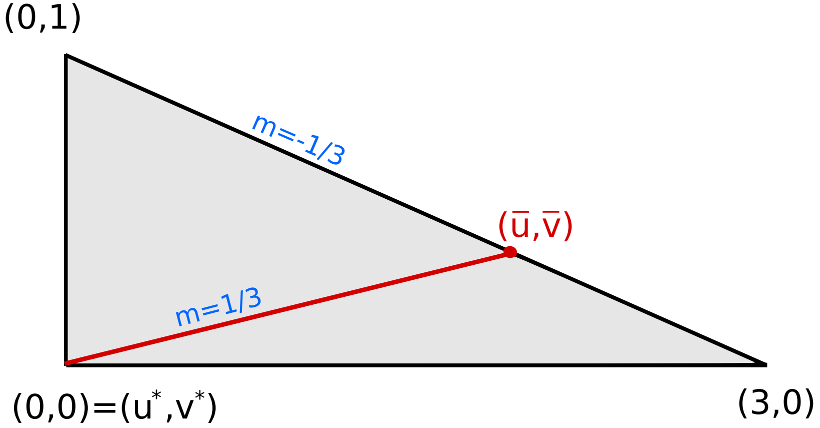

Nash Bargaining Example 3

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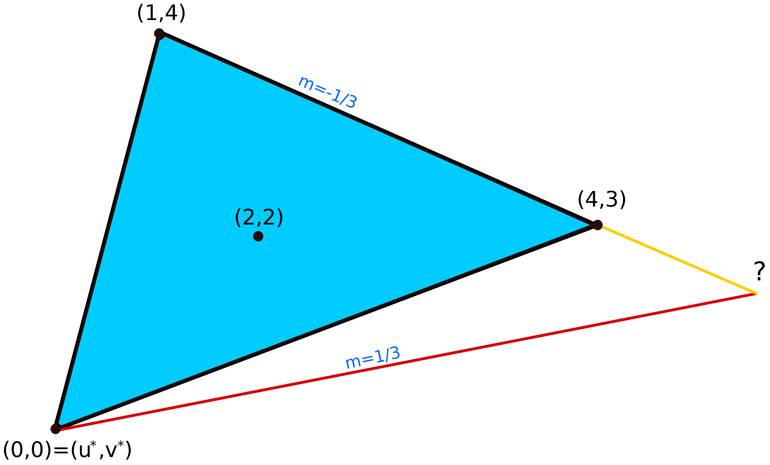
For the bimatrix game

$$\begin{pmatrix} (4,3) & (0,0) \\ (2,2) & (1,4) \end{pmatrix}$$

find the NTU solution if the threat point is $(0,0)$.

Nash Bargaining Example 3



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This means that uv is an increasing function along the upper right boundary of the triangle. So the max occurs at the corner $(4,3)$.

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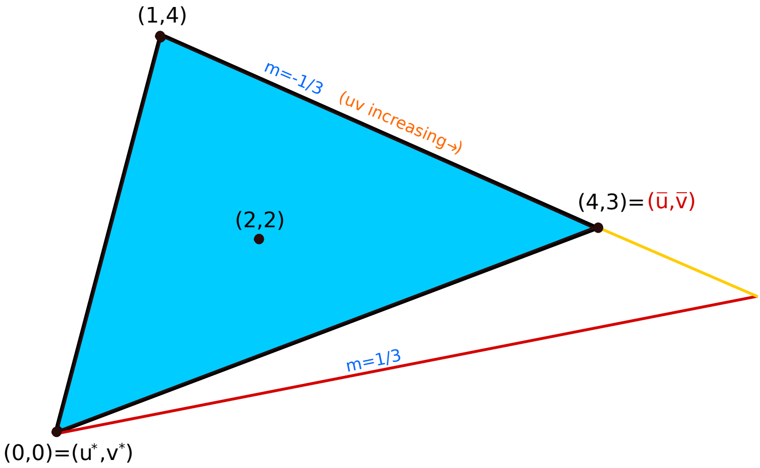

NTU: Alternative Approach

- We have seen that there is a simple closed form solution for the TU problem for bimatrix games.
- We have seen that finding an NTU solution is harder: each of our three examples used slightly different approached, and took some work.
- Sometimes we get lucky: if the TU solution to a game is in the NTU feasible set, then we are done already.
- **Example**. Consider the game

$$\begin{pmatrix} (2,0) & (0,0) \\ (0,0) & (0,2) \end{pmatrix}$$

The TU solution is $(1,1)$, which is in the NTU feasible set. So the NTU solution is $(1,1)$.

λ-Transfer Approach

- An alternative approach to NTU games follows from observing:
  - If the TU solution is in the NTU feasible set, we are done.
  - In an NTU game, we can rescale utility for an individual player.
- Given a bimatrix game (A, B), we scale one player's payoffs and TU solve this new game. If the solution is in the NTU feasible set, we have the NTU solution too.
- Then we take this solution and rescale the payoffs back to the original amount.

λ-Transfer Approach

- Let (A, B) be a given bimatrix game. We scale Player I's payoffs by λ, for some λ to be chosen later. The new game is $(\lambda A, B)$.
- Recall that the TU solution to this game is given by

$$\left(\frac{\sigma(\lambda)+\delta(\lambda)}{2}, \frac{\sigma(\lambda)-\delta(\lambda)}{2}\right)$$

  where $\sigma(\lambda)$ is the largest entry in the matrix $\lambda A + B$ and $\delta(\lambda)$ is the value of $\lambda A - B$.
- We vary λ until this solution is in the NTU feasible set of $(\lambda A, B)$.
- Then the NTU solution to (A, B) is given by

$$\left(\frac{\sigma(\lambda)+\delta(\lambda)}{2\lambda}, \frac{\sigma(\lambda)-\delta(\lambda)}{2}\right)$$

- It was shown by Lloyd Shapley that there is a unique value of λ that will work (1967).

λ-Transfer Approach

Summary

For a bimatrix game (A, B), let λ be chosen so that

$$\left(\frac{\sigma(\lambda)+\delta(\lambda)}{2\lambda}, \frac{\sigma(\lambda)-\delta(\lambda)}{2}\right)$$

is in the NTU feasible set of (A, B) (where $\sigma(\lambda)$ is the largest entry in the matrix $\lambda A + B$ and $\delta(\lambda)$ is the value of $\lambda A - B$). Then this is the NTU solution for (A, B).

Remark. This is a more formulaic approach to NTU problems. But that does not mean it is easy!

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Step 1: Find $\delta(\lambda)$:

We have

$$\lambda A - B = \begin{pmatrix} 5\lambda - 2 & 0 \\ 0 & \lambda - 4 \end{pmatrix}$$

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Checking if each of the four entries is a saddle point, we find that 0 is a saddle if $2/5 \leq \lambda \leq 4$. So when λ is in this range, $\delta(\lambda) = 0$.

(Note: If we don't find a solution in this range of λ, we will come back to this step and try other values of λ).

λ-Transfer Approach Example

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We have

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So $\sigma = \max(5\lambda + 2, \lambda + 4)$. If $\lambda \geq 1/2$, then $\sigma = 5\lambda + 2$.

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So when $1/2 \leq \lambda \leq 4$,

$$\left(\frac{\sigma(\lambda) + \delta(\lambda)}{2\lambda}, \frac{\sigma(\lambda) - \delta(\lambda)}{2} \right) = \left(\frac{5\lambda + 2}{2\lambda}, \frac{5\lambda + 2}{2} \right)$$

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The curve

$$\left(\frac{5\lambda+2}{2\lambda}, \frac{5\lambda+2}{2}\right), \quad \text{where } 1/2 \leq \lambda \leq 4$$

does intersect this line, at the point $(4.5, 2.25)$. This is the NTU solution.

Course Overview

- Part I: Combinatorial Games (Nim, Takeaway Games, Sprague-Grundy) ✓
- Part II: Two Person Zero-Sum Games (Matrix Games, Extensive and Strategic Forms) ✓
- Part III: Two Person General-Sum Games (Non-Cooperative, TU and NTU Cooperative) ✓
- Part IV: Many Person Games

Many-Person TU Games

- We will now consider Many-Person games.
- Payoffs are now a list of n numbers, where n is the number of players.
- We can describe such a game in extensive form or strategic form. Strategic form is no longer a matrix (entries require more than 2 numbers to index), so this form is a little unwieldy.
- We will introduce a new representation called the coalitional form.

Coalitional Form

- Consider a game with n players, who we will identify with the set $N = \{1, 2, \cdots, n\}$.
- **Def.** A coalition is a set S of players (ie. a subset of N). The set of all coalitions is denoted 2^N. The coalition with no players ($S = \emptyset$) is the empty coalition. The coalition with all players ($S = N$) is the grand coalition.
- For example, if $n = 2$ then the set of all coalitions is

$$2^N = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, N\}$$

Coalitional Form

- **Def.** The coalitional form of an n-person game is given by the pair (N, v) where $N = \{1, 2, \cdots, n\}$ is the set of players and v is a real-valued function $v : 2^N \to \mathbb{R}$, called the characteristic function of the game. v satisfies
  (i) $v(\emptyset) = 0$, and
  (ii) if S and T are disjoint coalitions ($S \cap T = \emptyset$) then $v(S) + v(T) \leq v(S \cup T)$.
- All this captures is the total payoff a group of players will earn if they form a team and compete against the remaining players.

Game Representations: Overview