Math 152: Applicable Mathematics and Computing

June 2, 2017

Announcements

- Office hour changes in Week 10: Josh's 9am Monday Office hours moved to Wednesday 2-3.
- If anyone has additional questions, I am happy to schedule additional office hours next week via email.
- I expect some TAs will change their hours for the exam, I will post any of these changes on the website.

Coalitional Form

- **Def.** The coalitional form of an n-person game is given by the pair (N, v) where $N = \{1, 2, \cdots, n\}$ is the set of players and v is a real-valued function $v : 2^N \to \mathbb{R}$, called the characteristic function of the game. v satisfies
  (i) $v(\emptyset) = 0$, and
  (ii) if S and T are disjoint coalitions ($S \cap T = \emptyset$) then $v(S) + v(T) \leq v(S \cup T)$.
- For each team (coalition) S, $v(S)$ is a measure of the total payoff of that team.

Strategic Form to Coalitional Form

- Given a game in strategic form, how do we convert this into coalitional form?
- Let $u_1, u_2, \cdots, u_n$ be the payoff functions for players $1, 2, \cdots, n$ in some game G with n players.
- To define a coalitional form, we need to specify what $v(S)$ is for every coalition S.
- We will define $v(S)$ to be the value of the 2-player zero-sum game obtained when the team S plays G against all other players. The payoff in this game is the sum of the payoffs to the players in S (we ignore the payoffs to all other players).

$$v(S) = \text{Val}\left(\sum_{i \in S} u_i\right)$$

It is easy to verify that this definition satisfies properties (i) and (ii).

Coalitional Form Example

Game

Three players simultaneously announces either 1 or 2. Let x be the sum of the announced numbers. If x is a multiple of 3, then Player 1 wins x and the remaining players win 0. If x is one more than a multiple of three, then Player 2 wins x and the other players win 0. Otherwise, Player 3 wins x and the others win 0.

What is the coalitional form of this game?

Coalitional Form Example

- There are 8 coalitions, we need to find v in each case.
- Easy ones first: the empty coalition receives no payoff, so $v(\emptyset) = 0$.
- The grand coalition will maximize the sum of profits to all players. So $v(N) = 6$, achieved if every player says '2'.
- Now consider the coalition $S = \{1\}$. We consider the zero-sum game where Player 1 competes with all remaining players.

$$\begin{array}{cc} & \begin{array}{cccc} (1,1) & (1,2) & (2,1) & (2,2) \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \left(\begin{array}{cccc} 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 \end{array} \right) \end{array}$$

By domination, the value is 0. So $v(\{1\}) = 0$.

Coalitional Form Example

- For $S = \{2\}$, we must solve the zero-sum game

$$\begin{array}{c c} & \begin{array}{cccc} (1,1) & (1,2) & (2,1) & (2,2) \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \left(\begin{array}{cccc} 0 & 4 & 4 & 0 \\ 4 & 0 & 0 & 0 \end{array} \right) \end{array}$$

  The value is 0, so $v(\{2\}) = 0$. (*Note that Player 2's strategies are* **rows** *here. The strategies for S will always be rows.*)

- For $S = \{3\}$, we must solve the zero-sum game

$$\begin{array}{c c} & \begin{array}{cccc} (1,1) & (1,2) & (2,1) & (2,2) \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \left(\begin{array}{cccc} 0 & 0 & 0 & 5 \\ 0 & 5 & 5 & 0 \end{array} \right) \end{array}$$

  The value is 0, so $v(\{3\}) = 0$.

Coalitional Form Example

- For $S = \{1, 2\}$, we have the matrix

$$\begin{array}{cc} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} (1,1) \\ (1,2) \\ (2,1) \\ (2,2) \end{array} & \begin{pmatrix} 3 & 4 \\ 4 & 0 \\ 4 & 0 \\ 0 & 6 \end{pmatrix} \end{array}$$

We can solve this via the graphical method. The value is $16/5$, so $v(\{1, 2\}) = 16/5$.

Coalitional Form Example

- For $S = \{1, 3\}$, we have the matrix

$$\begin{array}{c|cc} & 1 & 2 \\ \hline \begin{array}{c}(1,1)\\(1,2)\\(2,1)\\(2,2)\end{array} & \multicolumn{2}{c}{\begin{pmatrix} 3 & 0 \\ 0 & 5 \\ 0 & 5 \\ 5 & 6 \end{pmatrix}} \end{array}$$

This matrix has a saddle point, so $v(\{1, 3\}) = 5$.

Coalitional Form Example

- For $S = \{2, 3\}$, we have the matrix

$$
\begin{array}{c c}
 & \begin{matrix} 1 & 2 \end{matrix} \\
\begin{matrix} (1,1) \\ (1,2) \\ (2,1) \\ (2,2) \end{matrix} & \begin{pmatrix} 0 & 4 \\ 4 & 5 \\ 4 & 5 \\ 5 & 0 \end{pmatrix}
\end{array}
$$

We can reduce this to a 2×2 and solve, giving $v(\{2, 3\}) = 25/6$.

Coalitional Form Example

In summary, we have found:

$$
\begin{aligned}
v(\emptyset) &= 0 \\
v(\{1\}) &= 0 \\
v(\{2\}) &= 0 \\
v(\{3\}) &= 0 \\
v(\{1,2\}) &= 16/5 \\
v(\{1,3\}) &= 5 \\
v(\{2,3\}) &= 25/6 \\
v(\{1,2,3\}) &= 6
\end{aligned}
$$

Based on this data, which player is most valuable, and which is least valuable?

Coalitional Form to Strategic Form

- We have just seen how to construct the coalitional form of a game.
- If we are given the coalitional form, can we turn this into a game in strategic form?
- We can, but not uniquely. Here is one way:
  1. Each player simultaneously announces the coalition S_i they wish to be part of (so $i \in S_i$).
  2. If every player in S_i also announced exactly the set S_i, then every player in S_i receives payoff $v(S_i)/|S_i|$.
  3. But if *any* player in S_i announces a different set, then Player i receives payoff $v(\{i\})$ (ie. they play by themselves).
- **Remark.** If we convert from strategic form to coalitional form, and then back to strategic form, *we lose information about the game* (just like when switching back and forth between extensive and strategic form).

Coalitional Form to Strategic Form Example

Question

Consider the game in strategic form

$$\begin{pmatrix} (0,2) & (4,1) \\ (2,4) & (5,4) \end{pmatrix}$$

Convert this into coalitional form and back. Does the resulting strategic form differ in any important way from the original strategic form?

Coalitional Form to Strategic Form Example

Question

Consider the game in strategic form

$$\begin{pmatrix} (0,2) & (4,1) \\ (2,4) & (5,4) \end{pmatrix}$$

Convert this into coalitional form and back. Does the resulting strategic form differ in any important way from the original strategic form?

As always, they empty coalition receives no payoff: $v(\emptyset) = 0$.
And the grand coalition receives the maximum total payoff,

$$v(N) = v(\{1,2\}) = 5 + 4 = 9$$

Coalitional Form to Strategic Form Example

Question

Consider the game in strategic form

$$\begin{pmatrix} (0,2) & (4,1) \\ (2,4) & (5,4) \end{pmatrix}$$

Convert this into coalitional form and back. Does the resulting strategic form differ in any important way from the original strategic form?

For $S = \{1\}$, the payoff matrix is

$$\begin{pmatrix} 0 & 4 \\ 2 & 5 \end{pmatrix}$$

The value is 2, so $v(\{1\}) = 2$.

Coalitional Form to Strategic Form Example

Question

Consider the game in strategic form

$$\begin{pmatrix} (0,2) & (4,1) \\ (2,4) & (5,4) \end{pmatrix}$$

Convert this into coalitional form and back. Does the resulting strategic form differ in any important way from the original strategic form?

Finally we consider $S = \{2\}$. As mentioned before, the strategies for S will be the *rows* (not columns as is normal for Player 2). The payoff matrix is

$$\begin{pmatrix} 2 & 4 \\ 1 & 4 \end{pmatrix}$$

The value is 2, so $v(\{2\}) = 2$.

Coalitional Form to Strategic Form Example

Question

Consider the game in strategic form

$$\begin{pmatrix} (0,2) & (4,1) \\ (2,4) & (5,4) \end{pmatrix}$$

Convert this into coalitional form and back. Does the resulting strategic form differ in any important way from the original strategic form?

So, we have found that the coalitional form is given by

$$\begin{aligned} v(\emptyset) &= 0 \\ v(\{1\}) &= 2 \\ v(\{2\}) &= 2 \\ v(\{1,2\}) &= 9 \end{aligned}$$

Coalitional Form to Strategic Form Example

Now we convert back to strategic form. The pure strategies for Player 1 are the two coalitions they can be part of:

$$X = \{\{1\}, \{1,2\}\}$$

Similarly for Player 2:

$$Y = \{\{2\}, \{1,2\}\}$$

So the strategic form will look like

$$\begin{array}{cc} & \begin{array}{cc} \{2\} & \{1,2\} \end{array} \\ \begin{array}{c} \{1\} \\ \{1,2\} \end{array} & \begin{pmatrix} (?,?) & (?,?) \\ (?,?) & (?,?) \end{pmatrix} \end{array}$$

Coalitional Form to Strategic Form Example

Recall our rule for deciding the payoffs:

1. Each player simultaneously announces the coalition S_i they wish to be part of (so $i \in S_i$).
2. If every player in S_i also announced exactly the set S_i, then every player in S_i receives payoff $v(S_i)/|S_i|$.
3. But if *any* player in S_i announces a different set, then Player i receives payoff $v(\{i\})$ (ie. they play by themselves).

That is:

$$\begin{array}{cc} & \begin{array}{cc} \{2\} & \{1,2\} \end{array} \\ \begin{array}{c} \{1\} \\ \{1,2\} \end{array} & \begin{pmatrix} (v(\{1\}), v(\{2\})) & (v(\{1\}), v(\{2\})) \\ (v(\{1\}), v(\{2\})) & (\frac{v(\{1,2\})}{2}, \frac{v(\{1,2\})}{2}) \end{pmatrix} \end{array}$$

Coalitional Form to Strategic Form Example

So the bimatrix we get after converting back to strategic form is

$$\begin{array}{cc} & \begin{array}{cc} \{2\} & \{1,2\} \end{array} \\ \begin{array}{c} \{1\} \\ \{1,2\} \end{array} & \begin{pmatrix} (2,2) & (2,2) \\ (2,2) & (4.5,4.5) \end{pmatrix} \end{array}$$

This matrix is symmetric, so does not favor either player.
The original bimatrix was:

$$\begin{pmatrix} (0,2) & (4,1) \\ (2,4) & (5,4) \end{pmatrix}$$

These are **very different** bimatrices. The original bimatrix is not symmetric, in fact it favors Player 2: the TU solution is $(3.5, 5.5)$.