Math 152: Applicable Mathematics and Computing

June 5, 2017

Announcements

- Dun's office hours on Thursday are extended, from 12.30 – 3.30pm (in SDSC East 294).
- My office hours on Wednesday are extended (1–3pm), and other days by appointment.

Last Time: Coalitional Form Example

Game

Three players simultaneously announces either 1 or 2. Let x be the sum of the announced numbers. If x is a multiple of 3, then Player 1 wins x and the remaining players win 0. If x is one more than a multiple of three, then Player 2 wins x and the other players win 0. Otherwise, Player 3 wins x and the others win 0.

Last time, we found the coalitional form of the following game:

$$v(\emptyset) = v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$$

$$v(\{1,2\}) = 16/5, v(\{1,3\}) = 5, v(\{2,3\}) = 25/6, v(\{1,2,3\}) = 6$$

Question. If they all cooperate and recieve a joint payoff of 6, how should they divide this up?

Shapley Value

- Consider any game in coalitional form, where N is the set of players and v is the characteristic function.
- We assume that the players all cooperate and receive joint payoff $v(N)$.
- Let the final payoff for Player i be denoted $\phi_i(v)$.
- Like Nash's approach to bargaining, Shapley proposed several axioms that the solution should satisfy, and showed that they uniquely determine a solution.

Shapley Axioms

1. **Efficiency**: $\sum_{i \in N} \phi_i(v) = v(N)$.
2. **Symmetry**: If i, j are such that $v(S \cup \{i\}) = v(S \cup \{j\})$ for all sets S not containing i and j, then $\phi_i(v) = \phi_j(v)$.
3. **Dummy Axiom**: If i is such that $v(S \cup \{i\}) = v(S)$ for all S not containing i, then $\phi_i(v) = 0$.
4. **Additivity**: If u, v are characteristic functions, then $\phi(u + v) = \phi(u) + \phi(v)$.

Shapley Value: Example 1

Question

Let $N = \{1, 2, 3\}$. Let v be the characteristic function

$$v(\emptyset) = v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{1, 3\}) = v(\{2, 3\}) = 0$$

$$v(\{1, 2\}) = v(\{1, 2, 3\}) = 1$$

Find the Shapley values for each player, $\phi_i(v)$.

Shapley Value: Example 1

Question

Let $N = \{1, 2, 3\}$. Let v be the characteristic function

$$v(\emptyset) = v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{1,3\}) = v(\{2,3\}) = 0$$

$$v(\{1,2\}) = v(\{1,2,3\}) = 1$$

Find the Shapley values for each player, $\phi_i(v)$.

Axiom 1 implies

$$\phi_1(v) + \phi_2(v) + \phi_3(v) = 1$$

Shapley Value: Example 1

Question

Let $N = \{1, 2, 3\}$. Let v be the characteristic function

$$v(\emptyset) = v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{1,3\}) = v(\{2,3\}) = 0$$

$$v(\{1,2\}) = v(\{1,2,3\}) = 1$$

Find the Shapley values for each player, $\phi_i(v)$.

Axiom 1 implies

$$\phi_1(v) + \phi_2(v) + \phi_3(v) = 1$$

Axiom 2 implies $\phi_1(v) = \phi_2(v)$, since $v(\emptyset \cup \{1\}) = v(\emptyset \cup \{2\})$ and $v(\{3\} \cup \{1\}) = v(\{3\} \cup \{2\})$.

Shapley Value: Example 1

Question

Let $N = \{1, 2, 3\}$. Let v be the characteristic function

$$v(\emptyset) = v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{1,3\}) = v(\{2,3\}) = 0$$

$$v(\{1,2\}) = v(\{1,2,3\}) = 1$$

Find the Shapley values for each player, $\phi_i(v)$.

Axiom 1 implies

$$\phi_1(v) + \phi_2(v) + \phi_3(v) = 1$$

Axiom 2 implies $\phi_1(v) = \phi_2(v)$, since $v(\emptyset \cup \{1\}) = v(\emptyset \cup \{2\})$ and $v(\{3\} \cup \{1\}) = v(\{3\} \cup \{2\})$.

Axiom 3 implies $\phi_3(v) = 0$ (check!).

Shapley Value: Example 1

Question

Let $N = \{1, 2, 3\}$. Let v be the characteristic function

$$v(\emptyset) = v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{1,3\}) = v(\{2,3\}) = 0$$

$$v(\{1,2\}) = v(\{1,2,3\}) = 1$$

Find the Shapley values for each player, $\phi_i(v)$.

So,

$$\begin{aligned} \phi_1(v) &= 1/2 \\ \phi_2(v) &= 1/2 \\ \phi_3(v) &= 0 \end{aligned}$$

Shapley Value: The game w_S

More generally:

Question

For any set of players N, and any $S \subseteq N$, let w_S be the characteristic function defined by

$$w_S(T) = \begin{cases} 0 & \text{if } S \not\subseteq T \\ 1 & \text{if } S \subset T \end{cases}$$

Find the Shapley values for each player, $\phi_i(v)$.

Applying axioms 1–3 just as above, we get

$$\phi_i(w_S) = \begin{cases} 0 & \text{if } i \notin S \\ \frac{1}{|S|} & \text{if } i \in S \end{cases}$$

Remark. The game w_S is simply the game where the team scores 1 if the team contains all players in S, and otherwise the team scores 0.

Shapley Value: The game cw_S

More generally still:

Question

For any set of players N, and any $S \subseteq N$, let cw_S be the characteristic function defined by

$$cw_S(T) = \begin{cases} 0 & \text{if } S \not\subseteq T \\ c & \text{if } S \subset T \end{cases}$$

Find the Shapley values for each player, $\phi_i(v)$.

Applying axioms 1–3 just as above, we get

$$\phi_i(cw_S) = \begin{cases} 0 & \text{if } i \notin S \\ \frac{c}{|S|} & \text{if } i \in S \end{cases}$$

Shapley Value: Example 2

Question

Let $N = \{1, 2, 3\}$. Let v be the characteristic function

$$v(\emptyset) = v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{1,3\}) = v(\{2,3\}) = 0$$

$$v(\{1,2\}) = 1$$

$$v(\{1,2,3\}) = 4$$

Find the Shapley values for each player, $\phi_i(v)$.

Shapley Value: Example 2

Question

Let $N = \{1, 2, 3\}$. Let v be the characteristic function

$$v(\emptyset) = v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{1,3\}) = v(\{2,3\}) = 0$$

$$v(\{1,2\}) = 1$$

$$v(\{1,2,3\}) = 4$$

Find the Shapley values for each player, $\phi_i(v)$.

Notice that we can write

$$v = w_{\{1,2\}} + 3w_{\{1,2,3\}}$$

Shapley Value: Example 2

Question

Let $N = \{1, 2, 3\}$. Let v be the characteristic function

$$v(\emptyset) = v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{1,3\}) = v(\{2,3\}) = 0$$

$$v(\{1,2\}) = 1$$

$$v(\{1,2,3\}) = 4$$

Find the Shapley values for each player, $\phi_i(v)$.

Notice that we can write

$$v = w_{\{1,2\}} + 3w_{\{1,2,3\}}$$

So

$$\phi(v) = \phi(w_{\{1,2\}}) + \phi(3w_{\{1,2,3\}}) = (1/2, 1/2, 0) + (1, 1, 1)$$

Shapley Values: General Approach

- We know the Shapley value for characteristic functions of the form w_S.
- Given a general characteristic function v, if we can write it as

$$v = \sum_{S \subseteq N} c_S w_S$$

for some constants c_S, then we can compute $\phi(v)$.
- We will show that we can *always* write v in this way, and so we can always compute ϕ.

Shapley Values: General Approach

Theorem

There exists a unique function ϕ satisfying the Shapley axioms.

Proof.

- By above, we just need to show that any v can be written uniquely as

$$v = \sum_{S \subseteq N} c_S w_S$$

  for some constants c_S.
- We are forced to choose $c_{\{i\}} = v(\{i\})$ for each singleton set.
- Now inductively set

$$c_T = v(T) - \sum_{\substack{S \subset T \\ S \neq T}} c_S$$

Shapley Values: General Approach

Theorem

There exists a unique function ϕ satisfying the Shapley axioms.

Proof continued...

- We have set

$$c_T = v(T) - \sum_{\substack{S \subset T \\ S \neq T}} c_S$$

- Now

$$\begin{aligned} \sum_{S \subseteq N} c_S w_S(T) &= \sum_{S \subseteq T} c_T = c_T + \sum_{\substack{S \subset T \\ S \neq T}} c_T \\ &= v(T) - \sum_{\substack{S \subset T \\ S \neq T}} c_T + \sum_{\substack{S \subset T \\ S \neq T}} c_T = v(T) \end{aligned}$$

Shapley Value: Example 3

Question

Recall the coalitional form for the example from last time:

$$v(\emptyset) = v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$$

$$v(\{1,2\}) = 16/5, v(\{1,3\}) = 5, v(\{2,3\}) = 25/6, v(\{1,2,3\}) = 6$$

Find the Shapley values for each player, $\phi_i(v)$.

Shapley Value: Example 3

Question

Recall the coalitional form for the example from last time:

$$v(\emptyset) = v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$$

$$v(\{1,2\}) = 16/5, v(\{1,3\}) = 5, v(\{2,3\}) = 25/6, v(\{1,2,3\}) = 6$$

Find the Shapley values for each player, $\phi_i(v)$.

We need to find the constants c_S. We begin with

$$c_{\{1\}} = v(\{1\}) = 0$$

Similarly

$$c_{\{2\}} = c_{\{3\}} = 0$$

Shapley Value: Example 3

Question

Recall the coalitional form for the example from last time:

$$v(\emptyset) = v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$$

$$v(\{1,2\}) = 16/5, v(\{1,3\}) = 5, v(\{2,3\}) = 25/6, v(\{1,2,3\}) = 6$$

Find the Shapley values for each player, $\phi_i(v)$.

Next, we get

$$c_{\{1,2\}} = v(\{1,2\}) - c_{\{1\}} - c_{\{2\}} = 16/5$$

Shapley Value: Example 3

Question

Recall the coalitional form for the example from last time:

$$v(\emptyset) = v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$$

$$v(\{1,2\}) = 16/5, v(\{1,3\}) = 5, v(\{2,3\}) = 25/6, v(\{1,2,3\}) = 6$$

Find the Shapley values for each player, $\phi_i(v)$.

Next, we get

$$c_{\{1,2\}} = v(\{1,2\}) - c_{\{1\}} - c_{\{2\}} = 16/5$$

$$c_{\{1,3\}} = v(\{1,3\}) - c_{\{1\}} - c_{\{3\}} = 5$$

$$c_{\{2,3\}} = v(\{2,3\}) - c_{\{2\}} - c_{\{3\}} = 25/6$$

Shapley Value: Example 3

Question

Recall the coalitional form for the example from last time:

$$v(\emptyset) = v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$$

$$v(\{1,2\}) = 16/5, v(\{1,3\}) = 5, v(\{2,3\}) = 25/6, v(\{1,2,3\}) = 6$$

Find the Shapley values for each player, $\phi_i(v)$.

Finally

$$\begin{aligned} c_{\{1,2,3\}} &= v(\{1,2,3\}) - c_{\{1,2\}} - c_{\{1,3\}} - c_{\{2,3\}} - c_{\{1\}} - c_{\{2\}} - c_{\{3\}} \\ &= 6 - 16/5 - 5 - 25/6 \\ &= -191/30 \end{aligned}$$

Shapley Value: Example 3

Putting this together, we get

$$v = \frac{16}{5} w_{\{1,2\}} + 5 w_{\{1,3\}} + \frac{25}{6} w_{\{2,3\}} - \frac{191}{30} w_{\{1,2,3\}}$$

Shapley Value: Example 3

Putting this together, we get

$$v = \frac{16}{5} w_{\{1,2\}} + 5 w_{\{1,3\}} + \frac{25}{6} w_{\{2,3\}} - \frac{191}{30} w_{\{1,2,3\}}$$

Finally, we have

$$\phi_1(v) = \frac{16}{5} \cdot \frac{1}{2} + 5 \cdot \frac{1}{2} + \frac{25}{6} \cdot 0 - \frac{191}{30} \cdot \frac{1}{3} = 1.97\bar{7}$$

Shapley Value: Example 3

Putting this together, we get

$$v = \frac{16}{5} w_{\{1,2\}} + 5 w_{\{1,3\}} + \frac{25}{6} w_{\{2,3\}} - \frac{191}{30} w_{\{1,2,3\}}$$

Finally, we have

$$\phi_1(v) = \frac{16}{5} \cdot \frac{1}{2} + 5 \cdot \frac{1}{2} + \frac{25}{6} \cdot 0 - \frac{191}{30} \cdot \frac{1}{3} = 1.97\bar{7}$$

$$\phi_2(v) = \frac{16}{5} \cdot \frac{1}{2} + 5 \cdot 0 + \frac{25}{6} \cdot \frac{1}{2} - \frac{191}{30} \cdot \frac{1}{3} = 1.56\bar{1}$$

Shapley Value: Example 3

Putting this together, we get

$$v = \frac{16}{5} w_{\{1,2\}} + 5 w_{\{1,3\}} + \frac{25}{6} w_{\{2,3\}} - \frac{191}{30} w_{\{1,2,3\}}$$

Finally, we have

$$\phi_1(v) = \frac{16}{5} \cdot \frac{1}{2} + 5 \cdot \frac{1}{2} + \frac{25}{6} \cdot 0 - \frac{191}{30} \cdot \frac{1}{3} = 1.97\bar{7}$$

$$\phi_2(v) = \frac{16}{5} \cdot \frac{1}{2} + 5 \cdot 0 + \frac{25}{6} \cdot \frac{1}{2} - \frac{191}{30} \cdot \frac{1}{3} = 1.56\bar{1}$$

$$\phi_3(v) = \frac{16}{5} \cdot 0 + 5 \cdot \frac{1}{2} + \frac{25}{6} \cdot \frac{1}{2} - \frac{191}{30} \cdot \frac{1}{3} = 2.46\bar{1}$$

Shapley Value Question

Typically the numbers aren't as bad as in the last question. For example:

Question

Consider the game in coalitional form with $N = \{1, 2, 3\}$ and characteristic function

$$v(\emptyset) = 0, v(\{1\}) = 1, v(\{2\}) = 0, v(\{3\}) = 1$$

$$v(\{1, 2\}) = 4, v(\{1, 3\}) = 3, v(\{2, 3\}) = 5, v(\{1, 2, 3\}) = 8$$

(**Remark**. At this point we have covered all material necessary for the questions in homework 7, including the ungraded ones.)

Shapley Value: An Explicit Formula

Theorem

The Shapley value satisfies

$$\phi_i(v) = \sum_{\substack{S \subset N \\ i \in S}} \frac{(|S|-1)!(n-|S|)!}{n!}\,(v(S) - v(S - \{i\}))$$

Since we have shown that the Shapley function is unique, to prove that this formula holds we just need to show it satisfies axioms 1 – 4.

Remark. A formula is always nice, but the recursive approach from before may be less error-prone (and allows the computation of *all* Shapley values at the same time).