Math 152: Applicable Mathematics and Computing

June 7, 2017

Announcements

- Exam is on Monday in Center 115 at 8am.
- Bring a blue book.
- Cheat-sheet is allowed (1 page, front and back, handwritten).
- Will be cumulative, with a focus on material since midterm 2. So make sure to pay attention to recent homeworks (solutions will be posted today).
- Material: All material from Midterms 1 and 2, all of Part III, and Part IV chapters 1 and 3 (excluding: III subsection 4.4 and IV subsection 3.4).

Today

- We are done with the material needed for the homeworks and exam.
- Today we will tidy up some loose ends, and give another example of Shapley values.

Last Time: w_S

- Last time we examined the following problem: given a coalitional form game v (which maps teams to $\mathbb{R}$), what is a "fair" value of each player?
- We denoted this value by $\phi_i(v)$, where i is the number of the player in question, and v is our game.
- Our solution revolved around a simple game w_S: we have a set of key players S, and for any team T the payoff was 1 if T contained all of our key players, and 0 otherwise.

Last Time: w_S

For example, consider $N = \{1, 2, 3\}$ and $S = \{1, 3\}$. Then w_S is the game given by:

T	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$w_s(T)$	0	0	0	0	0	1	0	1

We saw that for such a game,

$$\phi_i(w_S) = \begin{cases} 0 & \text{if } i \notin S \\ \frac{1}{|S|} & \text{if } i \in S \end{cases}$$

Last Time: General Approach

- Our general solution: given *any* game v, write it as a linear combination of games of the form w_S.
- Then use additivity to find $\phi_i(v)$.
- For example:

T	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(T)$	0	0	0	0	1	0	0	4

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$w_{\{1,2\}}(T)$	0	0	0	0	1	0	0	1

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$3w_{\{1,2,3\}}(T)$	0	0	0	0	0	0	0	3

From this table, we can see that $v = w_{\{1,2\}} + 3w_{\{1,2,3\}}$. So

$$\phi_1(v) = \phi_1(w_{\{1,2\}}) + \phi_1(3w_{\{1,2,3\}}) = \frac{1}{2} + 3 \cdot \frac{1}{3} = 1.5$$

Shapley Value: Last Example

Question

Consider the game in coalitional form with $N = \{1, 2, 3\}$ and characteristic function

$$v(\emptyset) = 0, v(\{1\}) = 1, v(\{2\}) = 0, v(\{3\}) = 1$$

$$v(\{1, 2\}) = 4, v(\{1, 3\}) = 3, v(\{2, 3\}) = 5, v(\{1, 2, 3\}) = 8$$

Find the Shapley values $\phi_i(v)$ for each player.

1. First we write v as a sum of w_S's.
2. Then we use the additivity axiom to find $\phi_i(v)$ for $i = 1, 2, 3$.

A Detail Swept Under the Rug

- Axiom 4 says that for any games v, w:

$$\phi_i(v + w) = \phi_i(v) + \phi_i(w)$$

- But we have also been using

$$\phi_i(v - w) = \phi_i(v) - \phi_i(w)$$

(where $v, w, v - w$ are games).

- Are we cheating?

A Detail Swept Under the Rug

Theorem

If $v, w, v - w$ are games, then

$$\phi_i(v - w) = \phi_i(v) - \phi_i(w)$$

Proof.

By axiom 4, we have

$$\phi_i(v) = \phi_i((v - w) + w) = \phi_i(v - w) + \phi_i(w).$$

Rearranging this gives us the identity we desire. □

Remark. Some authors take this as an axiom (sometimes full linearity is assumed). As we can see, we do not need it as an axiom.

Exercise. Show that $\phi_i(cv) = c\phi_i(v)$ for any rational $c > 0$, just using our 4 axioms.

An Aside: Functional Equations

- Note the similarities between Nash's approach to bargaining and Shapley's approach here: define a set of rules our solution will follow, and find the function that satisfies the rules.
- These sorts of problems are known generally as functional equations (not to be confused with the area of *functional analysis*).
- These problems are nice, but are not taught commonly outside of mathematics competitions.
- **Example.** Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy $f(x + y) = f(x) + f(y)$.
  What if we remove the continuity assumption?
- A useful tool to have: clearly Nash and Shapley had it (and both were Putnam competitors).

Shapley Value: An Explicit Formula

Theorem

The Shapley value satisfies

$$\phi_i(v) = \sum_{\substack{S \subset N \\ i \in S}} \frac{(|S|-1)!(n-|S|)!}{n!} \left(v(S) - v(S - \{i\})\right)$$

- Since we have shown that ϕ is a unique function, we just need to verify the 4 axioms.
- We will go through these verifications shortly.
- First let us try to interpret it, and use this interpretation to get a nice probabilistic algorithm for computing Shapley values.

Shapley Value: An Explicit Formula

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$$\phi_i(v) = \sum_{\substack{S \subset N \\ i \in S}} \frac{(|S|-1)!(n-|S|)!}{n!} \left(v(S) - v(S - \{i\})\right)$$

Interpretation: Repeatedly form the grand coalition by randomly adding players in, one at a time. Each time we add in Player 1, record how much the score increases $(v(S) - v(S - \{1\}))$.
The average contribution of Player 1 will be exactly $\phi_1(v)$.

Shapley Value: Approximation Algorithm

Algorithm

Set some constant T, the number of steps.

1. Repeat steps 2–4 for T times, and output the **average** result:
2. Randomly add players one-by-one until we have all players.
3. Let S be the set of players in our team at the point we add Player i.
4. Record $v(S) - v(S - \{i\})$.

Assuming every iteration is independent, we get convergence by the Law of Large Numbers.

(**Remark.** These sorts of randomized approximation algorithms are known as *Monte Carlo* algorithms).

Shapley Value: An Explicit Formula

Proof of axiom 4 ($\phi_i(v+w) = \phi_i(v) + \phi_i(w)$).

$$\begin{aligned}
&\phi_i(v+w) \\
&= \sum_{\substack{S \subset N \\ i \in S}} \frac{(|S|-1)!(n-|S|)!}{n!} \left((v+w)(S) - (v+w)(S-\{i\})\right) \\
&= \sum_{\substack{S \subset N \\ i \in S}} \frac{(|S|-1)!(n-|S|)!}{n!} \left(v(S) - v(S-\{i\}) + w(S) - w(S-\{i\})\right) \\
&= \phi_i(v) + \phi_i(w)
\end{aligned}$$

□

Shapley Value: An Explicit Formula

Proof of axiom 3 (dummy axiom).

Assume that $v(S \cup \{i\}) = v(S)$ for all S. Then

$$\begin{aligned}
\phi_i(v) &= \sum_{\substack{S \subset N \\ i \in S}} \frac{(|S|-1)!(n-|S|)!}{n!}\,(v(S) - v(S - \{i\})) \\
&= \sum_{\substack{S \subset N \\ i \in S}} \frac{(|S|-1)!(n-|S|)!}{n!}\,(0) \\
&= 0
\end{aligned}$$

□

Shapley Value: An Explicit Formula

Proof of axiom 2 (symmetry).

Assume that $v(S \cup \{i\}) = v(S \cup \{j\})$ for all S not containing i or j. Then if S contains both i and j we have

$$v(S) - v(S - \{i\}) = v(S) - v(S - \{j\})$$

And if S contains i but not j, let $S' = S - \{i\} + \{j\}$. Then

$$v(S) - v(S - \{i\}) = v(S') - v(S' - \{j\})$$

With these two identities, we can check that $\phi_i(v) = \phi_j(v)$ in our formula. □

Shapley Value: An Explicit Formula

Proof of axiom 1 ($\sum \phi_i(v) = v(N)$).

We need to show

$$\sum_{i=1}^{n} \sum_{\substack{S \subset N \\ i \in S}} \frac{(|S|-1)!(n-|S|)!}{n!} \left(v(S) - v(S - \{i\})\right) = v(N)$$

Hint: Separate left-hand side into two sums, one with $v(S)$ and one with $v(S - \{i\})$. Now rewrite each double-sum as a single sum over sets $S \subset N$. □