Math 152: Applicable Mathematics and Computing

April 20, 2017

Announcements

Office hours:

- An-Vy Hoang on Mondays and Wednesdays from 11AM-1PM in APM 6436.
- Dun Qiu on Tuesdays and Thursdays from 1.30PM-3.30PM in SDSC E294.
- Nimish Srivastava on Mondays 3PM-5PM and Wednesdays 2PM-4PM in APM 2313.
- Josh Tobin on Mondays at 9AM and 1PM-3PM, and Wednesdays 1PM-2PM in APM 5768.

Remark about N-positions and P-positions

Remark. We have seen that if we know whether a position is an N or P-position, then we know who will win (if both players play optimally).

But it is important to notice that knowing the label of a position also tells you *how* to win: if you are in an N-position, you can win the game as long as you always put your opponent into a P-position on their turn.

Nim and nim-sim

The connection between nim and nim-sum is given by the following theorem:

Theorem (Bouton, 1902)

A position $(x_1, x_2, \cdots, x_k)$ in a game of Nim is a P-position if and only if

$$x_1 \oplus x_2 \oplus \cdots \oplus x_k = 0$$

Proof of Bouton's theorem

Set $\mathcal{P}$ be the positions $(x_1, x_2, \cdots, x_k)$ where

$$x_1 \oplus x_2 \oplus \cdots \oplus x_k = 0$$

Let $\mathcal{N}$ be all other positions. We want to show that:

1. Terminal positions are in $\mathcal{P}$.
2. From every position in $\mathcal{N}$, there is a move to a position in $\mathcal{P}$.
3. From every position in $\mathcal{P}$, every move is goes to a position in $\mathcal{N}$.

Proof of Bouton's theorem: (1)

We want to show that: **Terminal positions are in $\mathcal{P}$.**

The terminal positions are the all-zero positions, $(0, 0, \cdots, 0)$. We want to show that these are in $\mathcal{P}$.

Well, from our properties earlier, we know

$$0 \oplus 0 \oplus 0 \oplus \cdots \oplus 0 = 0$$

So $(0, 0, \cdots, 0) \in \mathcal{P}$.

Proof of Bouton's theorem: (2)

We want to show that: **From every position in $\mathcal{N}$, there is a move to a position in $\mathcal{P}$.**

Take some position $(x_1, x_2, \cdots, x_k) \in \mathcal{N}$. We just need to find a single move from this position to a position in $\mathcal{P}$.

We know that

$$x_1 \oplus x_2 \oplus \cdots \oplus x_k \neq 0$$

Write this nim-sum as a column addition, and find the first column whose sum is 1.

Let x_i be one of the pile sizes that has a 1 in column i. There must be at least one such pile, since the sum in the ith column was 1.

Now our winning move is to reduce pile i to be a size where the ith binary digit is 0, and all later binary digits are chosen to make the nim-sum equal to zero.

Proof of Bouton's theorem: (3)

We want to show that: **From every position in $\mathcal{P}$, every move is goes to a position in $\mathcal{N}$.**

Take a position $(x_1, x_2, \cdots, x_k) \in \mathcal{P}$. Then

$$x_1 \oplus x_2 \oplus x_3 \oplus \cdots \oplus x_k = 0$$

Every move from here simply makes one pile smaller, let's say the first pile is reduced from size x_1 to x_1', where $x_1' < x_1$.
By the cancellation property, we have

$$x_1' \oplus x_2 \oplus x_3 \oplus \cdots \oplus x_k \neq x_1 \oplus x_2 \oplus x_3 \oplus \cdots \oplus x_k$$

In particular,

$$x_1' \oplus x_2 \oplus x_3 \oplus \cdots \oplus x_k \neq 0$$

So $(x_1', x_2, \cdots, x_k) \in \mathcal{N}$.

Misère Nim

Question

Consider Nim played with the Misère rule. Determine the N- and P-positions in this case.

Misère Nim

- As usual, we first consider positions with a small number of chips.
- What if there is just one pile with n chips? N-position for $n \geq 2$. P-position for $n = 1$.
- What if there are many piles, each with just one chip? N-position for an even number of piles, P-position for an odd number of piles.]

Misère Nim

- So, we can win Misère Nim if we force our opponent to a position where there are no piles with more than 1 chip, and there are an odd number of piles with exactly 1 chip.
- If you examine the Bouton strategy for normal Nim, you can see that the winning player will be the player to reduce all piles to fewer than 2 chips.
- **Conclusion:** We can follow the Bouton strategy until there is **exactly** one pile with more than 1 chip. Then we can force our opponent to lose, by leaving an odd number of piles of size exactly 1.

Staircase Nim

Game (Staircase Nim)

In Staircase Nim, there is a stairway with piles of coins on some of the steps. On a player's turn, they choose some step, and any number of coins on that step, and move it to step immediately below. When the coins reach the bottom-most step, they are removed from the game.

Show that the P-positions in this game are exactly the positions where the nim-sum of the numbers of coins on the *odd* numbered steps is zero.