# Math 152: Applicable Mathematics and Computing

April 10, 2017

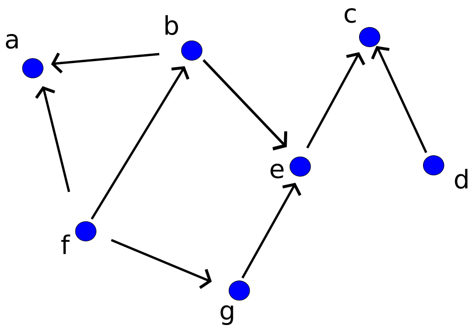
# Announcements

- Don't forget, first homework is due on Wednesday. Each TA has their own drop-box. Please provide justification for every answer.
- Change in office hours: Dun Qiu's Tuesdays office hours are now from 10.30am-12.30pm in SDSC E294.
- This week we will cover chapters 3 and 4 from Part I, and maybe parts of chapter 5 on Friday (if there is time). This will be the material that will be on the Week 3 exam.

# Directed Graphs

**Def.** A directed graph $G$ is a pair $(X, F)$ where $X$ is a nonempty set of *vertices* and $F$ is a function that for each vertex $x \in X$ gives a set $F(x)$ that is contained in $X$. $F(x)$ are called the *followers* of $x$.

This is a mathematical representation of a diagram like the following:



# Progressively Bounded Graphs

**Def.** A path of length $k$ is a list of vertices $x_0, x_1, x_2, \cdots, x_k$, such that for each $i \geq 1$, $x_i \in F(x_{i-1})$. (Note: this is the computer science definition of "path". In pure mathematics, this is usually called a "walk" instead.)

**Def.** A graph is progressively bounded, if for each vertex $x$ there is a constant $n$ such that every path starting at $x$ has length at most $n$.

# Graph Games

## Game (Graph Games)

Given a progressively bounded directed graph $G = (X, F)$, one can play a combinatorial game on the graph as follows. The positions of the game are the vertices in $X$. On a player's turn, if the current position is $x$, then the set of available moves is exactly $F(x)$, the followers of $x$.
As usual, in normal play, the last player who takes a move is the winner.

# Graph Games Example

- Each of the impartial combinatorial games we have seen so far can be written as graph games.
- **Board example.** Consider the subtraction game where each player may remove 1 or 2 coins on their turn. Given that there are at most 10 coins in total, represent this as a graph game, and identify the N and P-positions.

# The mex function

**Def.** Given a set of non-negative integers $X$, the mex of minimal excludant is the smallest non-negative integer that does not belong to $X$. For example

- $\text{mex}\{0, 1, 3, 4, 10, 11\} = 2$.
- $\text{mex}\{0, 1, 2, 3, 4\} = 5$.
- If $E$ is the set of even numbers $E = \{0, 2, 4, \cdots\}$, then $\text{mex}(E) = 1$.

# The Sprague-Grundy Function

**Def.** The Sprague-Grundy function of a directed graph $(X, F)$ is a function $g : X \to \mathbb{N}$ mapping vertices to non-negative integers, defined recursively by

$$g(x) = \operatorname{mex}\{g(y) : y \in F(x)\}$$

# The Sprague-Grundy Function

**Lemma**

Given a graph game $G$, the P-positions are exactly the positions $x$ where the Sprague-Grundy function $g(x)$ is 0.

**Proof.** Let $\mathcal{P}$ be the set of positions with Sprague-Grundy function equal to zero, and let $\mathcal{N}$ be all other positions.

- Then the terminal positions have no followers, so have SG value zero (so are in $\mathcal{P}$).
- Positions in $\mathcal{N}$ has SG value greater than zero. So it has a follower with SG value 0, so there is a move to a position in $\mathcal{P}$.
- Positions in $\mathcal{P}$ have SG value zero. Then all of its followers have Sprague-Grundy value strictly greater than zero, so all of the moves are to positions in $\mathcal{N}$.

# Sprague-Grundy Values of Games

We have seen that our impartial combinatorial games can be represented as graph games.
Every graph game has a Sprague-Grundy value.
So we can talk about Sprague-Grundy values of impartial combinatorial games.

# Algorithm for Sprague-Grundy Values of Games

Given an impartial combinatorial game, we can compute the Sprague-Grundy values of the positions of this game using the following algorithm.

1. For all terminal positions $x$, $g(x) = 0$.
2. Find any position $x$ for which we have found the Sprague-Grundy value for all of its followers.
3. Compute the mex of the SG values of followers of $x$. The answer is $g(x)$.
4. Repeat step 2.

**Board example.** Consider the subtraction game where each player may remove 1, 2 or 3 coins on their turn. Find the Sprague-Grundy values of every position of this game. What if you can remove 1, 3, 4 instead?

# Sprague-Grundy Motivation

So we have seen that if we know the Sprague-Grundy function of a game, we know the N-positions and P-positions. But why is this useful? We already had a way to find N and P-positions that involved less work. Of course, if there was no use, it wouldn't have been studied by Sprague and Grundy. We will see its importance later this week.