# Math 152: Applicable Mathematics and Computing

April 12, 2017

# Sprague-Grundy Example

**Question**

Consider a game where there is a single pile of chips on a table, and on your turn you may remove at least half of the chips. Which positions are P-positions? Determine the Sprague-Grundy function for all positions.

# Remark about Section 3.4

- We won't cover section 3.4 from Part I of the textbook. This concerns games which are not progressively bounded.
- Consider a game where there is some position $x$ which has an infinite number of followers, and for every integer $n \in \mathcal{N}$, there is a follower of $x$ with Sprague-Grundy value $n$. What is the mex?
- To answer this you need ordinals, which isn't covered by any of our prerequisite courses.

# Sum of Two Directed Graphs

**Def.** Given two progressively bounded directed graphs, $G_1 = (X_1, F_1)$ and $G_2 = (X_2, F_2)$, the **(disjunctive) sum** $G_1 + G_2$ of these two games is defined as follows. The vertex set is the set

$$X = X_1 \times X_2 = \{(x_1, x_2) : x_1 \in X_1, x_2 \in X_2\}.$$

The followers are defined by:

$$F(x_1, x_2) = (F_1(x_1) \times \{x_2\}) \cup (\{x_1\} \times F(x_2)).$$

# Sum of Two Games

- We have defined the sum of two directed graphs. We saw in the last lecture that every impartial combinatorial game can be written as a graph game. So this means we have defined the sum of two impartial combinatorial games.
- Informally, the sum of two games $G_1$ and $G_2$ is a game $G_1 + G_2$ where:
  1. A position in $G_1 + G_2$ consists of a position in $G_1$ and a position in $G_2$.
  2. On a player's turn, they can choose either $G_1$ and $G_2$, and make a move in the game they choose.
  3. The last player to make a move in *either* game wins. That is, a position $(x, y)$ is a terminal position exactly when $x$ is a terminal position in $G_1$ and $y$ is a terminal position in $G_2$.

# Sum of $n$ Games

If we can add two games, we can add any number of games. Eg. to add three games $G_1 + G_2 + G_3$, we add the game $G_1$ to the game $G_2 + G_3$. In practice, the resulting game looks like:

1. A position in $G_1 + G_2 + \cdots + G_k$ consists of a list of positions $(x_1, x_2, \cdots, x_k)$ where $x_i$ is a position in $G_i$.
2. On a player's turn, they can choose any game from $G_1, G_2, \cdots, G_k$ and make a move in that game.
3. The last player to make a move in *any* game wins. That is, a position $(x_1, x_2, \cdots, x_k)$ is a terminal position exactly when $x_i$ is a terminal position in $G_i$ for all $i$.

# Sprague–Grundy Theorem

**Theorem (Sprague–Grundy)**

If $g_i$ is the Sprague-Grundy function of $G_i$, $i = 1, 2, \cdots, n$, then $G = G_1 + G_2 + \cdots + G_n$ has Sprague–Grundy function

$$g(x_1, x_2, \cdots, x_n) = g_1(x_1) \oplus g_2(x_2) \oplus \cdots \oplus g_n(x_n)$$

# Sprague–Grundy Theorem and Nim

- We have already seen sums of games before: a game of Nim with multiple piles is the sum of single-pile Nim games.
- The Sprague–Grundy function of a single-pile Nim game is $g(x) = x$.
- We saw that the Sprague–Grundy function for Nim is given by

$$g(x_1, x_2, \cdots, x_k) = x_1 \oplus x_2 \oplus \cdots \oplus x_k$$

This agrees with Bouton's theorem.

# Sprague–Grundy Theorem and Nim

**Game (Subtraction game $\oplus$ Nim)**

If $G_1$ is the subtraction game where each player can subtract 1 or 2 coins on their turn, and $G_2$ is a Nim game with several piles. Let $G = G_1 + G_2$.

Notice that in the above game, we can treat the subtraction game as a Nim pile with $0, 1$ or 2 chips. We can do this for any game where we can compute the Sprague–Grundy function: in this way, the Sprague–Grundy function converts any impartial game into Nim.

# Sprague–Grundy Theorem Example

**Game (Google Interview Question)**

Consider a game where given a string consisting of the two characters '+' and '-', and on a player's turn they can turn any two adjacent '-' characters into two '+' characters. The last player to move is the winner.

# Sprague–Grundy Theorem Example

**Game (Google Interview Question, translated)**

Consider a game where there are several piles of chips. On a player's turn, they may take any pile which has at least two chips, remove two chips and (if they wish) split the remaining chips into two new piles. The last player to make a move is the winner.