Math 152: Applicable Mathematics and Computing

April 13, 2017

Sprague–Grundy Theorem

Theorem (Sprague–Grundy)

If g_i is the Sprague-Grundy function of G_i, $i = 1, 2, \cdots, n$, then $G = G_1 + G_2 + \cdots + G_n$ has Sprague–Grundy function

$$g(x_1, x_2, \cdots, x_n) = g_1(x_1) \oplus g_2(x_2) \oplus \cdots \oplus g_n(x_n)$$

Sprague–Grundy proof

Theorem (Sprague–Grundy)

Let G_1, G_2 be two impartial combinatorial games, with Sprague–Grundy functions g_1, g_2. Then $G = G_1 + G_2$ has Sprague–Grundy function

$$g(x_1, x_2) = g_1(x_1) \oplus g_2(x_2)$$

Proof. Let $b = g_1(x_1) \oplus g_2(x_2)$. We want to show that this is the SG value of the position (x_1, x_2). To prove this, we need to show two things:

1. No follower of (x_1, x_2) has g equal to b.
2. For any non-negative $a < b$, there is a follower of (x_1, x_2) with g equal to a.

Sprague–Grundy proof (1)

No follower of (x_1, x_2) **has** g **equal to** b:
A follower of (x_1, x_2) takes the form (x_1', x_2) or (x_1, x_2'), where $x_1 \neq x_1'$ and $x_2 \neq x_2'$.
Note that $g_1(x_1') \neq g_1(x_1)$, since x_1' is a follower of x_1. Then by the cancellation property

$$g_1(x_1') \oplus g_2(x_2) \neq g_1(x_1) \oplus g_2(x_2) = b$$

Similarly for any follower of the form (x_1, x_2').

Sprague–Grundy proof (2)

For any non-negative $a < b$, there is a follower of (x_1, x_2) with g equal to a:
Let k be the leftmost column where b is different to a. Since $a < b$, in this column a is 0 and b is 1.
Since $b = g_1(x_1) \oplus g_2(x_2)$, there is a 1 in column k of either $g_1(x_1)$ or $g_2(x_2)$. Say it is in $g_1(x_1)$.
$a \oplus b$ is zero in every column left of k, and 1 in column k. So

$$a \oplus b \oplus g_1(x_1) < g_1(x_1)$$

There is a move in G_1 to a position x_1' with

$$g_1(x_1') = a \oplus b \oplus g_1(x_1)$$

Then

$$g(x_1', x_2) = g_1(x_1') \oplus g_2(x_2) = a \oplus b \oplus g_1(x_1) \oplus g_2(x_2) = a \oplus b \oplus b = a.$$

Sprague–Grundy Theorem Example

Game (Google Interview Question)

Consider a game where given a string consisting of the two characters '+' and '-', and on a player's turn they can turn any two adjacent '-' characters into two '+' characters. The last player to move is the winner.

Sprague–Grundy Theorem Example

Game (Google Interview Question, translated)

Consider a game where there are several piles of chips. On a player's turn, they may take any pile which has at least two chips, remove two chips and (if they wish) split the remaining chips into two new piles. The last player to make a move is the winner.

Sprague–Grundy Theorem Example

Game (Google Interview Question, translated)

Consider a game where there are several piles of chips. On a player's turn, they may take any pile which has at least two chips, remove two chips and (if they wish) split the remaining chips into two new piles. The last player to make a move is the winner.

What are the followers of (6) in this game? (4), (3, 1) and (2, 2).
So

$$\begin{aligned} g(6) &= \text{mex}\,\{g(4), g(3,1), g(2,2)\} \\ &= \text{mex}\,\{g(4), g(3) \oplus g(1), g(2) \oplus g(2)\} \end{aligned}$$

Take-and-Break Games

Game (Lasker's Nim)

There are piles of chips on a table. On a player's turn, they select a pile and either remove any positive number of coins from that pile (like Nim), or they may split that pile into two smaller piles (without removing any coins). Find the Sprague–Grundy values for a single-pile version of this game.