# Math 152: Applicable Mathematics and Computing

April 16, 2017

# Announcements

- Please bring a blue book for the midterm on Friday. Some students will be taking the exam in Center 201, will announce which students before Wednesday's class.
- Exam covers Part I (chapters 1-4). Use the homeworks and lecture notes as a guide.

# Two-Player Zero Sum Games

- As before, we will be concerned with **two player** games.
- In particular we will study zero sum games: these are games where what one player wins is exactly what the other player loses.
- For example two-player poker: your winnings are exactly my losses.

# Two-Person Zero Sum Games

**Def.** A two-person zero sum game is a game with two players, which we will call Player I and Player II, where one player wins what the other player loses.

Eg. If Player I wins 5 dollars, this means that Player II loses 5 dollars. The prize-money is called the payoff.

# Two-Player Zero Sum Games

- Zero sum games are nice mathematically, because we can represent the outcome of the game as a *single* number $x$.
  $x$ represents the winnings of **Player I**.
- For example, if $x$ is 100 dollars, player I has taken 100 dollars from player II.
- But if $x$ is $-100$ dollars, player II has taken 100 dollars from player I.

# Strategic Form

**Def.** The strategic form of a two-person zero sum game is given by the triplet $(X, Y, A)$, where

1. $X$ is a nonempty set, called the strategies of Player I
2. $Y$ is a nonempty set, called the strategies of Player II
3. $A$ is a function mapping $X \times Y$ to $\mathbb{R}$ (ie. for each $x \in X$ and $y \in Y$, $A(x, y)$ is a real number). This represents the payoff, given the strategies of the players.

This is a mathematical way to represent a two-person zero sum game.
**Board example.** Write Rock-Paper-Scissors in strategic form, where the winner wins 1, and both players receive 0 in the case of a draw.

# Strategic Form of a Game

- We imagine the game being played in the following way: simultaneously, player I chooses her strategy $x$ from $X$ and player II chooses his strategy $y$ from $Y$. Both players do not know what the other player chooses.
- At the same moment, both players announce what strategy they picked.
- The players then consult $A(x, y)$ to see who wins, and the winner pays the loser (remember that a positive number means Player II pays Player I, if $A(x, y)$ is negative, then Player I pays Player II).

# Pure and Mixed Strategies

**Def.** The elements of the player's strategy sets $X$ and $Y$ are called pure strategies. These strategies involves no randomness.

**Def.** A mixed strategy is a random combination of pure strategies. For example, a player's strategy might consist of choosing pure strategy $x_1$ with probability $1/4$ and another pure strategy $x_2$ with probability $3/4$.

# Strategic Form Example II

**Game (Even/Odd)**

At the same time, both players will say either "one" or "two". These two numbers will be added together, if the sum is odd then player I wins, otherwise player II wins. The winner receives $x$ dollars, where $x$ is the sum of the two numbers chosen.

In this case each player only has two strategies: $X = \{1, 2\}$ and $Y = \{1, 2\}$.

The outcomes are:

$$\begin{array}{c} \\ x=1 \\ x=2 \end{array} \begin{array}{c} \begin{array}{cc} y=1 & y=2 \end{array} \\ \begin{pmatrix} -2 & +3 \\ +3 & -4 \end{pmatrix} \end{array}$$

(this is called the payoff matrix, which is a nice way to represent $A(x, y)$).

# Strategic Form Example: Even/Odd

- Player I has an advantage in this game. For example, here is one approach where, on average, Player I will not lose money:
- With probability 3/5, player I picks "one", and with probability 2/5 she picks "two". (This is a mixed strategy).
- If player II calls "one": then with probability 3/5 player II loses 2 dollars; with probability 2/5 player II wins 3. On average:

$$-2(3/5) + 3(2/5) = 0$$

- If player II calls "two": then with probability 3/5 player II wins 3 dollars; with probability 2/5 player II loses 4. On average:

$$3(3/5) - 4(2/5) = 1/5$$

- So on average player I can only win money, not lose money. In fact player I can do even better than this.

# Minimax Theorem

**Theorem (Minimax)**

For every two-person zero sum game where the sets $X$ and $Y$ are finite,

(1) there is a number $V$, called the value of the game,

(2) there is a mixed strategy for Player I such that I's average gain is at least $V$ no matter what II does, and

(3) there is a mixed strategy for Player II such that II's average loss is at most $V$ no matter what I does.

**Def.** A game is fair if $V = 0$, otherwise it is unfair.

**Goal.** We want a way to find the value of a game, given the payoff matrix, and the corresponding mixed strategy.

# Even/Odd: Optimal Play

- We return to the Even/Odd example.
- Let's try to find a way for player I to always win a positive amount, on average, no matter what player II does. We just need to decide on what probability $p$ to choose 1.
- To simplify things, let us try to find a $p$ so that player I's average winnings is the same *no matter what* player II does.
- Such a strategy is called an equalizing strategy. It does not exist for every game.

# Even/Odd: Optimal Play

- Let $p$ be the probability that player I chooses "one".
- If player II selects "one", player I wins on average

$$-2p + 3(1 - p) = -5p + 3$$

- If player II selects "two", player I wins on average

$$3p - 4(1 - p) = 7p - 4$$

- In an equalizing strategy, these are equal, so:

$$-5p + 3 = 7p - 4$$

- Solving these yields $p = 7/12$. In this case,

$$-5p + 3 = 7p - 4 = 1/12$$

That is, player I wins on average $1/12$ dollars *no matter what player II does*.

# Even/Odd: Optimal Play

- We have seen that in the even/odd game player I has a way to ensure she wins $1/12$ dollars on average.
- Similarly, player II has a strategy that ensures he loses no more than $1/12$ on average (to see this, repeat the computation from the previous slide form the perspective of player II). This is the *value* of this game.
- Because of player I's advantage, this is an unfair game.

# Strategic Form: Not so restrictive

- It seems that games in strategic form are very restrictive. Both players appear to only take a single turn.
- Actually many real, complicated games fit this form. For example, chess, tic-tac-toe, go, etc.
- For example, tic-tac-toe. The strategies for player I, consist of a list of all possible moves that player II can make, and what player I does in response.
- If both players choose such a strategy before the game starts, the outcome is determined without playing the game.

# Example 2: Equal

## Game (Equal)

Each player picks either "one" or "two". If both players say the same number, the player I wins $x$ dollars, where $x$ is the number both players chose. If both players say different numbers, the player II wins whatever player II said.

# Definition Review

- Zero–Sum game: Player I's gain is equivalent to Player II's loss.
- Pure strategy: An explicit description of what the player should do in all eventualities.
- Mixed strategy: A random combination of pure strategies.
- Equalizing strategy: A strategy where the player's average gain is the same no matter what the opposing player does.
- Value of a game: A number $V$ such that Player I has a strategy that wins at least $V$ on average, and Player II has a strategy such that Player II loses no more than $V$ on average.
- Unfair game: A game with value $V \neq 0$.