# Math 152: Applicable Mathematics and Computing

April 19, 2017

# Announcements

- If your last name / family name begins with the letter "Z" (as listed on TED), **take the midterm in Center 201** on Friday.
- Don't forget a bluebook!
- Homework solutions 1 already posted, homework solutions 2 will be posted after the deadline today.

# Strategic Form

**Def.** The strategic form of a two-person zero sum game is given by the triplet $(X, Y, A)$, where

1. $X$ is a nonempty set, called the strategies of Player I
2. $Y$ is a nonempty set, called the strategies of Player II
3. $A$ is a function mapping $X \times Y$ to $\mathbb{R}$ (ie. for each $x \in X$ and $y \in Y$, $A(x, y)$ is a real number). This represents the payoff, given the strategies of the players.

# Strategic Form: Not so restrictive

- It seems that games in strategic form are very restrictive. Both players appear to only take a single turn.
- Actually many real, complicated games can be written in strategic form. For example, chess, tic-tac-toe, go, etc.
- For example, tic-tac-toe. The strategies for a player consist of a list of all possible game positions and the move that they will make in that position.
- If both players choose such a strategy before the game starts, the outcome of the game is completely determined.

# Minimax Theorem

The Minimax theorem tells us there is a number $V$ (the value) such that:

- There is a strategy for Player I so that they win at least $V$ no matter what Player II does.
- There is a strategy for Player II so that they lose at most $V$ no matter what Player I does.

Given these strategies, they must be the optimal strategies for both players. Our goal is to be able to find these strategies and $V$.

# Equalizing Strategy

**Def.** An equalizing strategy is a mixed strategy such that our average winnings are the same for any strategy that our opponent may choose. Such a strategy does not always exist. For example, consider the game

described by the payoff matrix below:

$$\begin{array}{c} \\ x=1 \\ x=2 \end{array} \begin{array}{c} \begin{array}{cc} y=1 & y=2 \end{array} \\ \begin{pmatrix} 0 & 2 \\ 1 & 3 \end{pmatrix} \end{array}$$

No matter what strategy Player I uses, Player I will win more if Player II chooses $y = 2$ than if Player II chooses $y = 1$. **No equalizing strategy for Player I**.

# Equalizing Strategy: Motivation

Why do we care about equalizing strategies?

If there exists an equalizing strategy for both players, then these must be the players' optimal strategies. So we can find the value.

But as we have said, this approach does not always work!

# Example 2: Equal

## Game (Equal)

Each player picks either “one” or “two”. If both players say the same number, the player I wins $x$ dollars, where $x$ is the number both players chose. If both players say different numbers, the player II wins whatever player II said.

# Matrix Notation

Matrices are more than just a convenient way to represent the payoff function. Matrix/vector multiplication will be very useful to us. Eg. consider the payoff matrix:

$$A = \begin{bmatrix} -1 & 2 \\ 0 & 1 \\ 2 & -1 \end{bmatrix}$$

**Remember**: rows correspond to Player I strategies, columns to Player II.

Player I has three pure strategies. So a mixed strategy for Player II can be written as a vector $\mathbf{p} = (p_1, p_2, p_3)^T$

# Matrix Notation

$$A = \begin{bmatrix} -1 & 2 \\ 0 & 1 \\ 2 & -1 \end{bmatrix}$$

Say Player I has strategy $\mathbf{p} = (p_1, p_2, p_3)^T$, and Player II has strategy $\mathbf{q} = (q_1, q_2)^T$.

Then $\mathbf{p}^T A$ is the vector whose entries are the average winnings of Player I, with the strategy $\mathbf{p}$.

$A\mathbf{q}$ is the vector whose entries are the average losses of Player II, with this strategy $\mathbf{q}$.

$\mathbf{p}^T A\mathbf{q}$ is a real number, the average winnings of Player I (or losses of Player II), when they follow these strategies.

# Matrix Notation

Put in this form, the Minimax theorem is equivalent to saying that for any payoff matrix $A$, there is a number $V$, and strategies $\mathbf{p}$, $\mathbf{q}$, such that:

The vector $\mathbf{p}^T A$ has entries all at least $V$, and the vector $A\mathbf{q}$ has entries all at most $V$.

An equalizing strategy is a strategy where $\mathbf{p}^T A$ is a vector whose entries are all equal.

# Harder Games

How do we solve the games whose payoff matrices are below?

$$A = \begin{bmatrix} 3 & -1 & 4 \\ -1 & 3 & 4 \\ 2 & 1 & 10 \end{bmatrix}$$

$$B = \begin{bmatrix} 3 & -1 & 2 \\ -1 & 3 & 2 \\ 4 & -2 & 2 \end{bmatrix}$$

$$C = \begin{bmatrix} 4 & 1 & -3 \\ 3 & 2 & 5 \\ 0 & 1 & 6 \end{bmatrix}$$

# Definition Overview

Here is a brief (informal) overview of all of the terminology so far:

- Zero–Sum game: Player I's gain is equivalent to Player II's loss.
- Pure strategy: An explicit description of what the player should do in all eventualities.
- Mixed strategy: A random combination of pure strategies.
- Equalizing strategy: A strategy where the player's average gain is the same no matter what the opposing player does.
- Value of a game: A number $V$ such that Player I has a strategy that wins at least $V$ on average, and Player II has a strategy such that Player II loses no more than $V$ on average.
- Unfair game: A game with value $V \neq 0$.