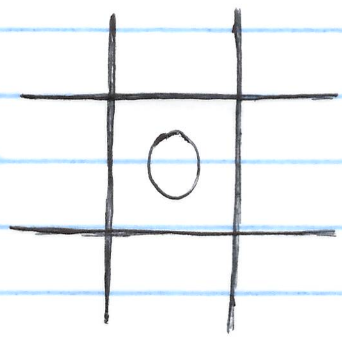
5. (a) The terminal positions are any positions where there are three 'O's in a single row, column or diagonal.

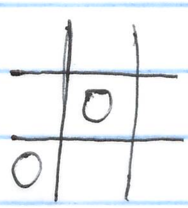
(b) To show that this position is an N-position, we just need to find one move from this position which moves to a P-position.

I claim that the position below, which we can move to from the position in the question, is a P-position:



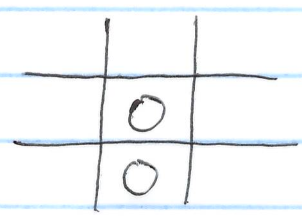
From this position, up to symmetry, there are two different types of moves:

(1)



and

(2)



Both of these are N-positions, since there is a clear winning move for the next player.

Hence the position

	O	

is a P-position.

It follows that the position in the question is an N-position.

(c) From the position given in the question, there are 8 different moves. 6 of these positions are immediately seen to be N-positions, as the next player can then complete three consecutive 'O's. For example,

O		
O		

is a N-position.

There are only two remaining positions to identify:

(1)

	O	
O		

and (2)

		O
O		

By symmetry, these are either both N-positions or both P-positions.

From position (1), all but one move allows the following player to immediately win. The remaining move is to the position

	O	
		O
O		

This is a P-position (again, any move from here allows the following player to win).
So (1) and (2) are N-positions.

Every move from the position in the question is to an N-position, so the position in the question is a P-position.

6. (a) If we flip the first H, the new position is:

(1) T T H T T

If we flip the second H, there are 3 possibilities (depending on whether or not we flip a coin to the left):

(2) H T T T T ← Just flip coin 3
(3) H H T T T ← flip coin 3 then coin 2
(4) T T T T T ← flip coin 3 · then coin 1.

(b) If there is a single H, on the next player's turn they must flip that H, and if they wish, flip any of the coins to the left. If they only flip the H, then the resulting position is the terminal position "TT...T", and so this player has won the game.

So, every position with exactly one "H" is an N-position.

This is just motivation. Not required as part of an answer.

(c) Consider a position with exactly one "H".
Eg. "TTTH". The available moves are:

(1) TTTT
(2) HTTT
(3) THTT
(4) TTHT.

Notice that, in effect, the moves are simply moving the "H" to the left. This is like a game of Nim with a single pile of 4 coins. This motivates the conjecture below.

Eg. for THTTH, $k=2$, $x_1=2$, $x_2=5$.

Conjecture: Let $x_1, x_2, \ldots, x_k$ be the positions of the "H" coins, where k is the number of "H" coins currently showing. Then this position is a P-position exactly when $x_1 \oplus x_2 \oplus \cdots \oplus x_k = 0$.

Pf. Let $\mathcal{P}$ be the set of positions with $x_1 \oplus \cdots \oplus x_k = 0$ and let $\mathcal{N}$ be all other positions.
We let the positions with no "H" be in $\mathcal{P}$ also. We need to show:

(1) Terminal positions are in $\mathcal{P}$
(2) From any position in $\mathcal{N}$ there is a move to a position in $\mathcal{P}$.
(3) From any position in $\mathcal{P}$, every move is to a position in $\mathcal{N}$.

(1) The terminal positions are those positions with no "H", which we have said are in $\mathcal{P}$.

(2) Take any position in $\mathcal{N}$. Say there are k "H"s, and they are located at $x_1, x_2, \dots, x_k$. By definition of $\mathcal{N}$, $x_1 \oplus x_2 \oplus \cdots \oplus x_k \neq 0$.

From the proof of Bouton's theorem, there is some x_i which we can decrease so that this nim-sum becomes zero. Assume, without loss of generality, that this is x_1: so there is some $x_1' < x_1$ so that

$$x_1' \oplus x_2 \oplus \cdots \oplus x_k = 0. \qquad (*)$$

Now, if there is a "T" currently in position x_1', then we can make the following move in Turning Turtles: flip the "H" in position x_1, and then flip the "T" in position x_1'. The resulting position is in $\mathcal{P}$, by $(*)$.

Otherwise, there is a "H" in position x_1'. This means that $x_j = x_1'$ for some j. Without loss of generality, $x_2 = x_1'$. Then:

$$x_1' \oplus x_2 \oplus x_3 \oplus \cdots \oplus x_k = 0$$

$$\Rightarrow \; 0 \oplus x_3 \oplus \cdots \oplus x_k = 0$$

$$\Rightarrow \quad x_3 \oplus x_4 \oplus \cdots \oplus x_k = 0 \qquad (**)$$

We can make the following move in Turning Turtles: flip the "H" in position x_1, and flip the "H" in position $x_1' = x_2$ (since $x_1' < x_1$).
By $(**)$, this position is in $\mathcal{P}$.

(3) Take any position in $\mathcal{P}$, with k "H"s, located at $x_1, x_2, x_3, \ldots, x_k$. A move in Turning Turtles either:

a move that flips only one "H"

(i) changes exactly one x_i, making it smaller; OR

a move that flips two "H"s

(ii) sets exactly two x_i, x_j to 0, where $x_i \neq x_j$

In the first case, say the new position has "H"s at:
$$x_1', x_2, \ldots, x_k,$$
where $x_1' < x_1$. Then

Cancellation property

$$x_1' \oplus x_2 \oplus \cdots \oplus x_k \neq x_1 \oplus x_2 \oplus \cdots \oplus x_k = 0.$$

In the second case, say the new position has "H"s at $x_3, x_4, \ldots, x_k$.

Cancellation property

Then $x_3 \oplus \cdots \oplus x_k \neq x_1 \oplus x_2 \oplus \cdots \oplus x_k = 0$,
since $x_1 \neq x_2$ (so $x_1 \oplus x_2 \neq 0$).
So every move is to a position in $\mathcal{N}$.