1. The terminal positions are $0, 1,$ so

$$g(0) = 0, \quad g(1) = 0.$$

$$g(2) = \text{mex}\{g(0)\} = 1.$$
$$g(3) = \text{mex}\{g(0), g(1)\} = \text{mex}\{0\} = 1.$$
$$g(4) = \text{mex}\{g(1), g(2)\} = \text{mex}\{0, 1\} = 2.$$
$$g(5) = \text{mex}\{g(2), g(3)\} = \text{mex}\{1\} = 0.$$
$$g(6) = \text{mex}\{g(3), g(4)\} = \text{mex}\{1, 2\} = 0.$$
$$g(7) = \text{mex}\{g(4), g(5)\} = \text{mex}\{0, 2\} = 1$$
$$g(8) = \text{mex}\{g(5), g(6)\} = \text{mex}\{0, 0\} = 1.$$
$$g(9) = \text{mex}\{g(6), g(7)\} = \text{mex}\{0, 1\} = 2.$$
$$g(10) = \text{mex}\{g(7), g(8)\} = \text{mex}\{1\} = 0.$$

So we have observed a repeating pattern:

$$\underline{0, 0, 1, 1, 2}, \ \underline{0, 0, 1, 1, 2}, \ \underline{0 \ldots}$$

Since $g(x)$ depends on only the 3 previous values, and this pattern has length 5 (>3), this pattern will continue (beginning again at each multiple of 5).

2. First we list the followers of every position:

$F(a) = \{\}$ $\quad F(d) = \{c\}$ $\quad F(g) = \{e\}.$

$F(b) = \{a, e\}$ $\quad F(e) = \{c\}$

$F(c) = \{\}$ $\quad F(f) = \{a, b, g\}$

So $SG(a) = \text{mex}\{\} = 0$

& $SG(c) = \text{mex}\{\} = 0.$

Then $SG(d) = \text{mex}\{SG(c)\} = \text{mex}\{0\} = 1$

& $SG(e) = \text{mex}\{SG(c)\} = \text{mex}\{0\} = 1.$

$SG(b) = \text{mex}\{SG(a), SG(e)\} = \text{mex}\{0, 1\} = 2.$

$SG(g) = \text{mex}\{SG(e)\} = \text{mex}\{1\} = 0.$

$SG(f) = \text{mex}\{SG(a), SG(b), SG(g)\} = \text{mex}\{0, 2\} = 1.$

3. The Sprague-Grundy function for G_1 is given by:

$$g_1(x) = \begin{cases} 0 & \text{if } 3 \text{ divides } x \\ 1 & \text{if } 3 \text{ divides } x-1 \\ 2 & \text{if } 3 \text{ divides } x-2. \end{cases}$$

This can be justified by computing g_1 for $x \le 6$, and noting that $g_1(x)$ depends only on $g_1(x-1)$ and $g_1(x-2)$.

For Nim, the Sprague-Grundy function g_2 is given by

$$\begin{aligned} g_2(x_1, x_2, \dots, x_k) &= g_2(x_1) \oplus \cdots \oplus g_2(x_k) \\ &= x_1 \oplus x_2 \oplus \cdots \oplus x_k. \end{aligned}$$

By the Sprague-Grundy Thm, the S.G. function g for $G_1 + G_2$ is given by:

$$g(x, (x_1, x_2, x_3)) = g_1(x) \oplus g_2(x_1, x_2, x_3)$$

For the given position, we have:

$$\begin{aligned} g(10, (1,6,7)) &= g_1(10) \oplus g_2(1,6,7) \\ &= 1 \oplus 1 \oplus 6 \oplus 7 \\ &= 1. \end{aligned}$$

$$\begin{array}{ccc l} 0 & 0 & 1 & \leftarrow (1) \\ 0 & 0 & 1 & \leftarrow (2) \\ 1 & 1 & 0 & \\ 1 & 1 & 1 & \leftarrow (3) \\ \hline 0 & 0 & 1 & \leftarrow \end{array}$$

From the row nim-sum above, 3 winning moves exist:

(1) Remove 1 coin from the game G_1

(2) Remove 1 coin from the 1-pile in G_2

(3) Remove 1 coin from the 7-pile in G_2.

4. The terminal positions in this game are any positions where all piles have ≤ 1 coin.

In particular,

$$g(0) = 0$$

$$g(1) = 0$$

Observation: In this game, like in Nim, a position where there are multiple piles of coins is the same as the sum-of-games of several single-pile games. This is true because every move in this game operates on a single pile only.

Consequence: $g(x,y) = g(x) \oplus g(y)$, where (x,y) is a position in this game with two piles, of sizes x & y.

Then:

$$g(2) = \text{mex}\{g(1,1)\} = \text{mex}\{g(1)\oplus g(1)\} = \text{mex}\{0\} = 1.$$

$$g(3) = \text{mex}\{g(2,1)\} = \text{mex}\{g(2)\oplus g(1)\} = \text{mex}\{1\} = 0.$$

$$g(4) = \text{mex}\{g(3,1), g(2,2)\} = \text{mex}\{g(3)\oplus g(1), g(2)\oplus g(2)\} = \text{mex}\{0\} = 1.$$

$$g(5) = \text{mex}\{g(4,1), g(3,2)\} = \text{mex}\{g(4)\oplus g(1), g(3)\oplus g(2)\} = \text{mex}\{1\} = 0.$$

$$g(6) = \text{mex}\{g(5)\oplus g(1), g(4)\oplus g(2), g(3)\oplus g(3)\} = \text{mex}\{0\} = 1.$$

$$g(7) = \text{mex}\{g(6)\oplus g(1), g(5)\oplus g(2), g(4)\oplus g(3)\} = \text{mex}\{1\} = 0.$$

$$g(8) = \text{mex}\{g(7)\oplus g(1), g(6)\oplus g(2), g(5)\oplus g(3), g(4)\oplus g(4)\} = \text{mex}\{0\} = 1.$$

$$g(9) = \text{mex}\{g(8)\oplus g(1), g(7)\oplus g(2), g(6)\oplus g(3), g(5)\oplus g(4)\} = \text{mex}\{1\} = 0$$

$$g(10) = \text{mex}\{g(9)\oplus g(1), g(8)\oplus g(2), g(7)\oplus g(3), g(6)\oplus g(4), g(5)\oplus g(5)\} = \text{mex}\{0\} = 1.$$