MA152 Solutions to Homework 3

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1. Firstly note that the third column dominates the second column. Removing the dominated column, we get

$$\begin{pmatrix} 5 & -2 \\ -3 & 5 \\ -2 & 6 \end{pmatrix}$$

Now the last row dominates the middle row, leaving

$$\begin{pmatrix} 5 & -2 \\ -2 & 6 \end{pmatrix}$$

There is no saddle point, and since this is a 2×2 matrix it means we can solve it by finding equalizing strategies.

Let $\mathbf{p} = (p \ \ 1-p)^T$ be Player I's strategy. The expected gain if Player II picks the first column is $5p - 2(1-p)$. The expected gain if Player II picks the second column is $-2p + 6(1-p)$. Setting these to be equal, we get

$$5p - 2(1-p) = -2p + 6(1-p) \Rightarrow p = 8/15$$

So the value of the game is $V = 5(8/15) - 2(7/15) = 26/15$.

Let $\mathbf{q} = (q \ \ 1-q)^T$ be Player II's strategy. The expected payoff is Player I picks the first row is $5q - 2(1-q)$. For the second row, the expected payoff is $-2q + 6(1-q)$. These are the same equations as before, so $q = 8/15$.

2. The payoff matrix in this case is

$$\begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \end{array}$$

This is a 2×2 matrix with no saddle point, so we look for an equalizing strategy. Let $\mathbf{p} = (p \;\; 1-p)^T$ be Player I's strategy. The payoff if Player II selects the first column is $p - 2(1-p)$. If Player II selects the second column, it is $-2p + 4(1-p)$. Setting these equal we get $p = 2/3$. The value of the game is $V = (2/3) - 2(1/3) = 0$.

Let Let $\mathbf{q} = (q \;\; 1-q)^T$ be Player II's strategy. The payoff across the first row is $q - 2(1-q)$, and across the second row is $-2q + 4(1-q)$. These are the same equations as before, so $q = 2/3$.

3. (a) There is a saddle point in this matrix: the entry 0. The corresponding optimal strategies are $\mathbf{p} = (1 \; 0)$ and $\mathbf{q} = (0 \; 1)$, (since the players will choose the row and column that contain the saddle point).

   (b) There is no saddle point in this matrix, so since it is a 2×2 matrix we know there will be a pair of equalizing strategies. Let $\mathbf{p} = (p \;\; 1-p)^T$ be Player I's strategy. The expected gain across the first column is $3p - (1-p)$ and across the second column is $5(1-p)$. Setting these equal and solving, we get $p = 2/3$, and $V = 3(2/3) - (1/3) = 5/3$.

   Let $\mathbf{q} = (q \;\; 1-q)^T$ be Player II's strategy. Then the expected payoff across the first row is $3q$ and across the second row is $-q + 5(1-q)$. Setting these equal and solving, we get $q = q = 5/9$.

4. This is question 5(a) in Part II Chapter 2 of the textbook. The solution is given on page II-2 of the online "Solutions to Exercises".

5. The payoff matrix is

$$\begin{array}{c} \\ 1 \\ -2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} -2 & 3 & -4 \\ 3 & -4 & 5 \\ -4 & 5 & -6 \end{pmatrix} \end{array}$$

Let $\mathbf{p}, \mathbf{q}$ be the optimal strategies for each player. By the assumption in the question, each entry of $\mathbf{q}$ is positive. By the principle of indifference, the payoff down each column is equal to V.

This gives

$$\begin{aligned}
-2p_1 + 3p_2 - 4p_3 &= V \\
3p_1 - 4p_2 + 5p_3 &= V \\
-4p_1 + 5p_2 - 6p_3 &= V \\
p_1 + p_2 + p_3 &= 1
\end{aligned}$$

Solving these equations yields $p_1 = 1/4$, $p_2 = 1/2$, $p_3 = 1/4$, $V = 0$.

These entries are all positive, so by the Principle of Indifference the expected payoff is the same no matter which row Player I chooses. So Player II has an equalizing strategy. Since the matrix is symmetric, the equations for this equalizing strategy are the same as those above, and hence $q_1 = 1/4$, $q_2 = 1/2$, $q_3 = 1/4$.

6. This is question 11 in Part II Chapter 2 of the textbook. The solution is given on page II-4 of the online "Solutions to Exercises".