MA152 Solutions to Homework 4

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1. We have
$$A^{-1} = \begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/3 \end{pmatrix}$$
So $\mathbf{1}^T A^{-1}\mathbf{1} = (1/2 + 1/3 + 1 + 1/3) = 13/6$. This is not zero, so we can use the nonsingular matrix theorem. The value of the game is $V = (\mathbf{1}^T A^{-1}\mathbf{1})^{-1} = 6/13$. The optimal strategies for each players are given by
$$\mathbf{p}^T = V\mathbf{1}^T A^{-1} = \begin{pmatrix} 3/13 & 2/13 & 6/13 & 2/13 \end{pmatrix}$$
and
$$\mathbf{q}^T = V\mathbf{1}A^{-1} = \begin{pmatrix} 3/13 & 2/13 & 6/13 & 2/13 \end{pmatrix}$$

2. This is an upper triangular matrix, so let us try looking for an equalizing strategy. Let Player I's strategy be $\mathbf{p}^T = \begin{pmatrix} p_1 & p_2 & p_3 & p_4 \end{pmatrix}$ and let Player II's strategy be $\mathbf{p}^T = \begin{pmatrix} q_1 & q_2 & q_3 & q_4 \end{pmatrix}$. We get the following system of equations for Player I:
$$\begin{aligned} p_1 \qquad\qquad\qquad &= V \\ -p_1 + \ p_2 \qquad\qquad &= V \\ -2p_2 + p_3 \qquad &= V \\ -p_1 + \ p_2 - p_3 + p_4 &= V \\ p_1 + \ p_2 + p_3 + p_4 &= 1 \end{aligned}$$

Solving this, we get $V = 1/13$ and

$$\mathbf{p}^T = \begin{pmatrix} 1/13 & 2/13 & 5/13 & 5/13 \end{pmatrix}$$

For Player II, we get the system

$$\begin{aligned} q_1 - q_2 \qquad - q_4 &= V \\ q_2 - 2q_3 + q_4 &= V \\ q_3 - q_4 &= V \\ q_4 &= V \\ q_1 + q_2 + q_3 + q_4 &= 1 \end{aligned}$$

Solving this system, we get

$$\mathbf{q}^T = \begin{pmatrix} 6/13 & 4/13 & 2/13 & 1/13 \end{pmatrix}$$

3. Denote the columns by A,B and C, and denote the rows by 1, 2 and 3. Now, swapping rows 1 and 3 and columns A and C leaves the payoff matrix unchanged. So, by Invariance, we can assume that the optimal strategies $\mathbf{p}$ and $\mathbf{q}$ satisfy $\mathbf{p}(1) = \mathbf{p}(3)$ and $\mathbf{q}(A) = \mathbf{q}(C)$.

   We look for an equalizing strategy. For Player I, we get

$$\begin{aligned} -4\mathbf{p}(1) + \mathbf{p}(2) + 2\mathbf{p}(3) &= V \\ \mathbf{p}(1) - 5\mathbf{p}(2) + \mathbf{p}(3) &= V \\ 2\mathbf{p}(1) + \mathbf{p}(2) - 4\mathbf{p}(3) &= V \\ \mathbf{p}(1) + \mathbf{p}(2) + \mathbf{p}(3) &= 1 \end{aligned}$$

   Now applying $\mathbf{p}(1) = \mathbf{p}(3)$, we get

$$\begin{aligned} -2\mathbf{p}(1) + \mathbf{p}(2) &= V \\ 2\mathbf{p}(1) - 5\mathbf{p}(2) &= V \\ 2\mathbf{p}(1) + \mathbf{p}(2) &= 1 \end{aligned}$$

Solving this system, wet get $V = -1/2$ and

$$\mathbf{p}^T = \begin{pmatrix} 3/8 & 1/4 & 3/8 \end{pmatrix}$$

Since the matrix is symmetric, the system of equations for finding an equalizing strategy for Player II is the same as for Player I. So we also have

$$\mathbf{q}^T = \begin{pmatrix} 3/8 & 1/4 & 3/8 \end{pmatrix}$$

4. *Answer 1:* Note that the payoff matrix A in this question satisfies $A(i, j) = i - j$. So we have

$$A(i, j) = i - j = -(j - i) = -A(j, i)$$

and so A is skew-symmetric. Therefore the value of the game is zero.

Answer 2: We can note that the bottom right entry of the matrix (entry $(1000, 1000)$) is a saddle point, since $A(1000, 1000) = 0$, and 0 is the smallest number in the last row and the largest number in the last column. So $V = 0$.

Answer 3: Note that the last row dominates all other rows. Similarly, the last column dominates all other columns. All that remains is the bottom right entry, which is 0. So $V = 0$.

5. (a) Given this strategy for Player II, the possible payoffs are

$$A\mathbf{q} = \begin{pmatrix} 2q + 1 \\ -4q + 5 \end{pmatrix}$$

Player I will choose the first row if $2q + 1 > -4q + 5 \Leftrightarrow q > 2/3$, and the second row otherwise.

(b) Player II knows that this is what Player I will choose. So Player II will choose the value of q that maximizes the outcome.

If $0 \leq q \leq 2/3$, then the payoff is $-4q + 5$. Along this interval, this function is minimized at $q = 2/3$. The payoff in this case is $7/3$.

If$2/3 < q \leq 1$, then the payoff is $2q + 1$, which is always greater than $7/3$ on this interval. So the minimum payoff that Player II can achieve is $7/3$, if they choose $q = 2/3$.

6. (a) Let C denote the matrix whose entries are all c. So $B = A + C$.

Answer 1: Let $\mathbf{p}, \mathbf{q}$ be strategies chosen by players I and II during a game of B. Then the payoff is

$$\mathbf{p}^T B\mathbf{q} = \mathbf{p}^T (A + C)\mathbf{q} = \mathbf{p}^T A\mathbf{q} + c$$

where the last equality follows from the fact that the entries of $\mathbf{p}$ and $\mathbf{q}$ sum to 1.

In particular, playing the game B is exactly the same as playing the game A, and then adding c to the payoff of A. It is clear that the optimal strategy for both players in such a game is just to use their optimal strategies for A. In this case the payoff is $V+c$, and so this is the value of B.

Answer 2: Let $\mathbf{p}, \mathbf{q}$ be a pair of optimal strategies for A. Then, from the definition, we get that

$$\min(\mathbf{p}^T A) = V = \max(A\mathbf{q})$$

(where min and max here represent the min and max entry of a vector).

Now we have

$$\min(\mathbf{p}^T B) = \min(\mathbf{p}^T (A + C)) = \min(p^T A + \mathbf{p}^T C)$$

Note that since $\mathbf{p}$ is a probability vector, every entry of the vector $\mathbf{p}^T C$ is just c. So

$$\min(\mathbf{p}^T B) = \min(\mathbf{p}^T A) + c = V + c$$

Similarly,

$$\max(B\mathbf{q}) = V + c$$

From the definition of value, we get that $V + c$ is the value of B, with optimal strategies $\mathbf{p}$ and $\mathbf{q}$.

(b) Let A be the matrix given in the question. Notice that

$$\begin{pmatrix} 2 & 1 & 0 & -1 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 \\ 5 & 4 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 0 & -1 & -2 & -3 \\ 1 & 0 & -1 & -2 \\ 2 & 1 & 0 & -1 \\ 3 & 2 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \end{pmatrix}$$

The first matrix on the right hand side is skew-symmetric, so has value 0. From (a), we get that

$$\mathrm{Val}(A) = 0 + 2 = 2$$