MA152 Solutions to Homework 5

May 18, 2017

1. First we list the pure strategies for each player. Player I has three pure strategies, which we can denote A, B and C. Player II's pure strategies each consist of a pair (since Player II has two information sets). For example, the strategy $(1, 4)$ for Player II denotes the strategy where Player II picks option 1 when in the information set on the left and option 4 when in either of the vertices of the information set on the right. A full list of Player II's strategies is: $(1, 3), (1, 4), (2, 3), (2, 4)$.

   Next we construct the payoffs. For each pair of pure strategies, one from each player, we need to compute the average payoff. For example, if Player I uses strategy A and Player II uses strategy $(2, 4)$, then on the first turn Player I will select option A, and on the next turn Player II will select option 2. Then with probability $1/3$ the payoff is $1/2$, and with probability $2/3$ the payoff is 0, so the average payoff is $\frac{1}{3}(1/2) + \frac{2}{3}(0) = 1/6$.

   Continuing in this way, and constructing the payoffs for all pairs, we get the following payoff matrix:

$$\begin{array}{c c} & \begin{array}{cccc} (1,3) & (1,4) & (2,3) & (2,4) \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{pmatrix} -1 & -1 & 1/6 & 1/6 \\ 1 & -1 & 1 & -1 \\ -2 & 3 & -2 & 3 \end{pmatrix} \end{array}$$

2.

   (1, I) In the first round, Player I picks **row 1**.

(2, II) Based on Player I's previous strategies, Player II guesses that Player I's strategy is always to pick row 1, ie. $\mathbf{p} = (1, 0)^T$. In this case, Player II wishes to pick **column 2**, for a payoff of -1. However, Player II might be wrong about Player I's strategy, in which case Player I may be using a better strategy with a better payoff. So we obtain a lower bound $V \geq -1$.

(3, I Based on Player II's previous strategies, Player I guesses that Player II's strategy is always to pick column 2, ie. $\mathbf{q} = (0, 1)^T$. In this case, Player I wishes to pick **row 2**, for a payoff of 3. However, Player I might be wrong about Player II's strategy, in which case Player II may be using a better strategy where Player II loses less. So we obtain an upper bound $V \leq 3$.

(4, II) Based on Player I's previous strategies, Player II guesses that Player I's strategy is $\mathbf{p} = (1/2, 1/2)^T$. In this case, the payoffs are $\mathbf{p}^T A = (1/2, 1)$. Player II wishes to pick **column 1**, for a payoff of 1/2. So we obtain a lower bound $V \geq 1/2$.

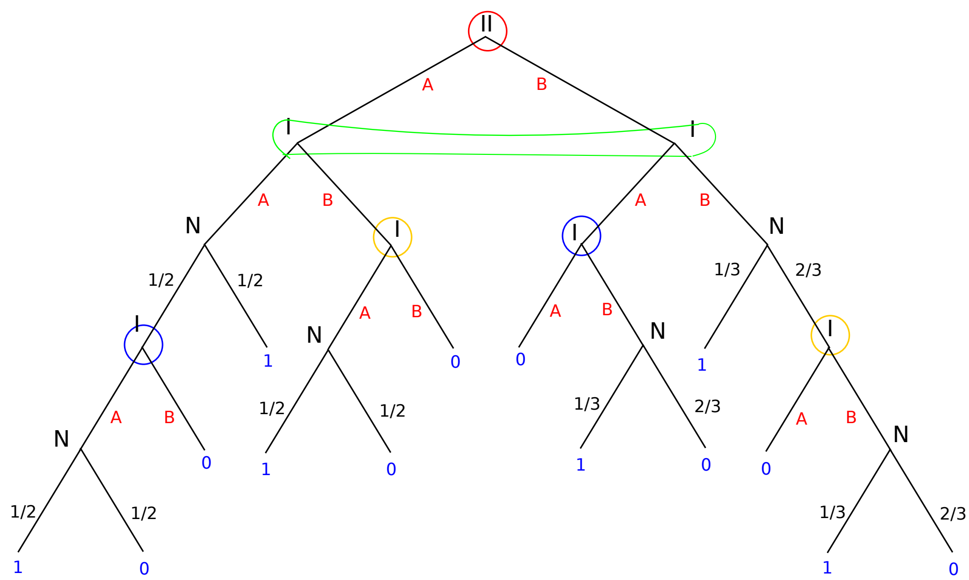
(5,I) Based on Player II's previous strategies, Player I guesses that Player II's strategy is $\mathbf{q} = (1/2, 1/2)^T$. In this case, the payoffs are $A\mathbf{q} = (0, 3/2)^T$. Player I wishes to pick **row 2**, for a payoff of 1/2. So we obtain an upper bound $V \leq 3/2$.

(6,II) Based on Player I's previous strategies, Player II guesses that Player I's strategy is $\mathbf{p} = (1/3, 2/3)^T$. In this case, the payoffs are $\mathbf{p}^T A = (1/3, 5/3)^T$. Player II wishes to pick **column 1**, for a payoff of 1/3. So we obtain an lower bound $V \geq 1/3$.

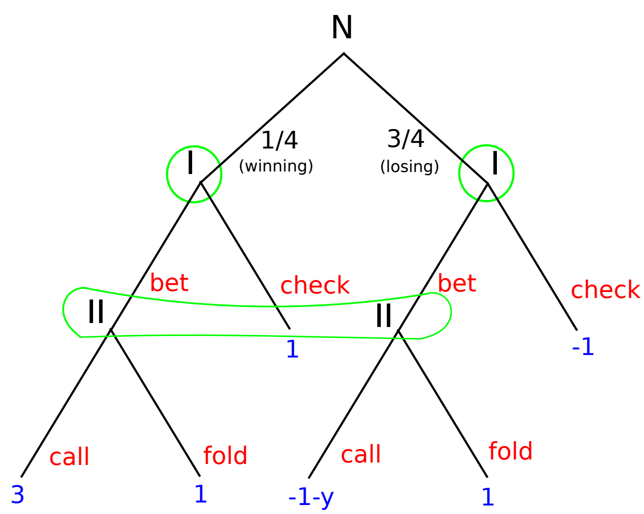
(5,I) Based on Player II's previous strategies, Player I guesses that Player II's strategy is $\mathbf{q} = (2/3, 1/3)^T$. In this case, the payoffs are $A\mathbf{q} = (1/3, 1)^T$. Player I wishes to pick **row 2**, for a payoff of 1. So we obtain an upper bound $V \leq 1$.

We now have the upper and lower bounds, $1/2 \leq V \leq 1$. The upper and lower limits differ by 1/2, as required by the question, so we stop.

3. The Extensive Form is given below. Note that the information sets here are specified by different colors: the two vertices that are in blue circles are in the same information set, and the two vertices in yellow circles are in the same information set.



4.



We wlll denote Player II's pure strategies by c and f. Player I has two information sets. Every pure strategy needs to give Player II's action in each of these information sets. For example, the strategy (b, c) is the strategy where Player I bets if they are dealt a winning card, and checks otherwise. The full list of Player I's strategies is (b, b), (b, c), (c, b), (c, c).

We compute the payoffs for every pair of strategies. For example, if Player I chooses strategy (b,b), and Player II chooses strategy c, then the average payoff is given by $\frac{1}{4}(3)+\frac{3}{4}(-1-y)=-\frac{3}{4}y$. The full payoff matrix is given by

$$\begin{array}{c} \\ (b,b) \\ (b,c) \\ (c,b) \\ (c,c) \end{array} \begin{array}{c} \begin{array}{cc} \quad c & \qquad f \end{array} \\ \begin{pmatrix} -\frac{3y}{4} & 1 \\ 0 & -1/2 \\ -\frac{1}{2}-\frac{3y}{4} & 1 \\ -1/2 & -1/2 \end{pmatrix} \end{array}$$

The first row dominates the third, and the second dominates the fourth. What remains is a 2×2 matrix without a saddle point. So we look for equalizing strategies. Let p be Player I's probability of choosing row 1. Then we get

$$\begin{aligned} \frac{-3y}{4}p &= V \\ p-\frac{1}{2}(1-p) &= V \end{aligned}$$

Solving this system, we find $V=-y/(4+2y)$.

5. (a) Player I's guess of Player II's strategy in this case is $\mathbf{q}=(1/3,2/3)^T$. The payoffs would then be $A\mathbf{q}=(\sqrt{2}/3,2/3)^T$. Since $2/3>\sqrt{2}/3$, Player I will pick the second row.

   (b) Let's say that, so far, Player II has picked column one x times and column two y times. Then Player I's guess at Player II's strategy is

$$\mathbf{q}=\begin{pmatrix} \frac{x}{x+y} & \frac{y}{x+y} \end{pmatrix}^T$$

   In this case the payoffs are

$$A\mathbf{q}=\begin{pmatrix} \frac{x}{x+y} & \frac{y}{x+y} \end{pmatrix}^T=\begin{pmatrix} \frac{\sqrt{2}x}{x+y} & \frac{y}{x+y} \end{pmatrix}^T$$

   Now if $\sqrt{2}x>y$, then Player I will choose row 1. Otherwise Player I will choose row 2. Now, Player II knows that this is what Player I will do. So Player II knows Player I's strategy in advance. If

Player I is going to choose row 1, Player II can choose column 2 and the payoff will be zero. If Player II is going to choose row 2, Player II can choose column 1 and the payoff will be zero. In either case, Player II can be sure that they will lose no money.

There is one possible complication: what if $\sqrt{2}x = y$? Then Player II can not predict what Player I will do. But this cannot happen, sincce x, y are integers, with at least one non-zero, and $\sqrt{2}$ is irrational.