MA152 Solutions to Homework 7

June 7, 2017

1. This is a textbook exercise, Part III section 4 question 1(a). The answer is in the online solutions.

2. The TU solution is given by

$$\left(\frac{\sigma+\delta}{2}, \frac{\sigma-\delta}{2}\right)$$

We have $\sigma = \max(A+B) = 7$, attained at the entry $(4,3)$. We have

$$\delta = \mathrm{Val}\begin{pmatrix} 2 & 1 & 0 \\ -5 & 1 & 5 \end{pmatrix}$$

Taking 1/2 of column 1 and 1/2 of column 3 we get a column that is less than or equal to column 2. So the middle column is dominated by the other two. Hence

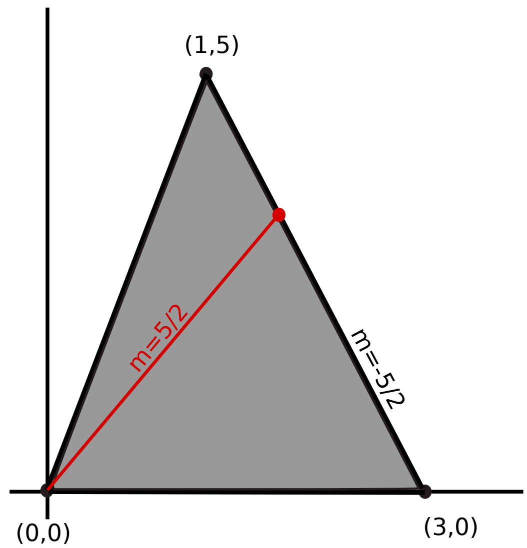
$$\delta = \mathrm{Val}\begin{pmatrix} 2 & 0 \\ -5 & 5 \end{pmatrix}$$

This is a 2×2 matrix without a saddle point, so we look for an equalizing strategy, We find that $\delta = 5/6$. Hence the TU solution is given by

$$\left(\frac{47}{12}, \frac{37}{12}\right)$$

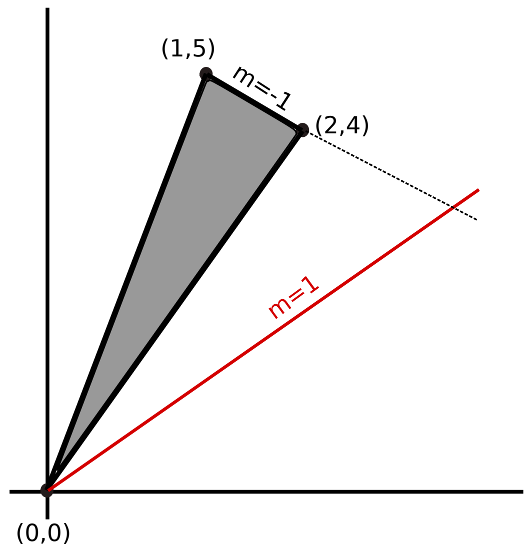
This is the result of the game *after* the sidepayment. We have said that the players will agree to choose strategies so that their payoff at the end of the game is $(4,3)$. To go from $(4,3)$ to $(47/12, 37/12)$, Player I must give a sidepayment of $1/12$ to Player II.

3. (a) The NTU feasible set is pictured below.



The red line is the line from the threat point $(0,0)$ with slope equal to the negative of the slope along the Pareto optimal curve. The solution is the intersection of the red line and the line segment between $(1,5)$ and $(3,0)$. The first line has equation $y = \frac{5}{2}x$ and the second line has equation $y = -\frac{5}{2}(x-3)$. The intersection is at $x = 3/2, y = 15/4$.

(b) The NTU feasible set is pictured below.



In this case, the line from the threat point $(0,0)$ with slope equal to the negative of the Pareto optimal curve does not intersect the Pareto optimal curve.

By the theorem of Nash, we know that the NTU solution is the point on the Pareto curve that maximizes xy. From the picture, we can deduce that along the Pareto optimal curve the function xy is increasing as we move from left to right. So the max occurs at $(2,4)$.

4. This is a textbook exercise, Part III section 4 question 4.

5. We have $c_{\{1\}} = v(\{1\}) = 1$, $c_{\{1\}} = v(\{2\}) = 0$ and $c_{\{3\}} = v(\{3\}) = 1$. Now

$$c_{\{1,2\}} = v(\{1,2\}) - c_{\{1\}} - c_{\{2\}} = 2 - 1 - 0 = 1$$
$$c_{\{1,3\}} = v(\{1,3\}) - c_{\{1\}} - c_{\{3\}} = 4 - 1 - 2 = 1$$
$$c_{\{2,3\}} = v(\{2,3\}) - c_{\{2\}} - c_{\{3\}} = 3 - 0 - 2 = 1$$
$$c_{\{1,2,3\}} = v(\{1,2,3\}) - c_{\{1\}} - c_{\{2\}} - c_{\{3\}} - c_{\{1,2\}} - c_{\{1,3\}} - c_{\{2,3\}} = 1$$

So

$$v = w_{\{1\}} + 2w_{\{3\}} + w_{\{1,2\}} + w_{\{1,3\}} + w_{\{2,3\}} + w_{\{1,2,3\}}$$

Then

$$
\begin{aligned}
\phi_1(v) &= \phi_1(w_{\{1\}}) + \phi_1(2w_{\{3\}}) + \phi_1(w_{\{1,2\}}) + \phi_1(w_{\{1,3\}}) + \phi_1(w_{\{2,3\}}) + \phi_1(w_{\{1,2,3\}}) \\
&= 1 + 0 + \frac{1}{2} + \frac{1}{2} + 0 + \frac{1}{3} = 2 + \frac{1}{3}
\end{aligned}
$$

$$
\begin{aligned}
\phi_2(v) &= \phi_2(w_{\{1\}}) + \phi_2(2w_{\{3\}}) + \phi_2(w_{\{1,2\}}) + \phi_2(w_{\{1,3\}}) + \phi_2(w_{\{2,3\}}) + \phi_2(w_{\{1,2,3\}}) \\
&= 0 + 0 + \frac{1}{2} + 0 + \frac{1}{2} + \frac{1}{3} = 1 + \frac{1}{3}
\end{aligned}
$$

$$
\begin{aligned}
\phi_3(v) &= \phi_3(w_{\{1\}}) + \phi_3(2w_{\{3\}}) + \phi_3(w_{\{1,2\}}) + \phi_3(w_{\{1,3\}}) + \phi_3(w_{\{2,3\}}) + \phi_3(w_{\{1,2,3\}}) \\
&= 0 + 2 + 0 + \frac{1}{2} + \frac{1}{2} + \frac{1}{3} = 3 + \frac{1}{3}
\end{aligned}
$$

6. This is the exercise Part IV section 1 question 4 in the textbook.

7. As always, we have $v(\emptyset) = 0$. For the coalition $S = \{1\}$, we consider the zero-sum game where 1 is playing against the remaining players $\{2, 3\}$. The payoff matrix in this case is

$$
\begin{array}{c c}
 & \begin{array}{cccc} (0,0) & (0,1) & (1,0) & (1,1) \end{array} \\
\begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}
\end{array}
$$

The left-column dominates the others, so the value of the game is 0. So $v(\{1\}) = 0$.

For $S = \{2\}$, the payoff matrix is

$$
\begin{array}{c c}
 & \begin{array}{cccc} (0,0) & (0,1) & (1,0) & (1,1) \end{array} \\
\begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}
\end{array}
$$

By domination, the value of this game is 0. So $v(\{2\}) = 0$. Similarly, we get $v(\{3\}) = 0$.

Now if $S = \{1, 2\}$, the payoff matrix is

$$
\begin{array}{c c}
 & \begin{array}{cc} 0 & 1 \end{array} \\
\begin{array}{c} (0,0) \\ (0,1) \\ (1,0) \\ (1,1) \end{array} & \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 3 \end{pmatrix}
\end{array}
$$

By domination, we can reduce this to

$$\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

The value of this game is 3/4. So $v(\{1,2\}) = 3/4$.

Now if $S = \{1,3\}$, the payoff matrix is

$$\begin{array}{c} \\ (0,0) \\ (0,1) \\ (1,0) \\ (1,1) \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{pmatrix} 0 & 0 \\ 0 & 2 \\ 0 & 2 \\ 2 & 3 \end{pmatrix} \end{array}$$

This game has a saddle point, so the value is 2. We get $v(\{1,3\}) = 2$.

If $S = \{2,3\}$, the payoff matrix is

$$\begin{array}{c} \\ (0,0) \\ (0,1) \\ (1,0) \\ (1,1) \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{pmatrix} 0 & 1 \\ 1 & 2 \\ 1 & 2 \\ 2 & 0 \end{pmatrix} \end{array}$$

The last two rows dominate the other, so the value is 4/3. We get $v(\{2,3\}) = 4/3$.

Finally, for $v(\{1,2,3\})$ we need to find the maximum possible payoff if all players cooperate. In this case, if they all say 1 the payoff is 3. So $v(\{1,2,3\}) = 3$.