1. The terminal positions in this game are: 0. So this is a P-position, and now we iteratively label positions, following the usual algorithm:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
P	N	P	N	P	N	N	N	N	N	N	P	N	P	N	P	N	N	N	N

20	21	22	23	...
N	N	P	N	...

The label of a position depends only on the six previous labels, and since we have seen a repeating pattern of length 11 ("PNPNPNNNNNN"), this repeating pattern will continue, restarting at positions which are multiples of 11.

2. The terminal position in this game is 0. The only position from which all moves ~~move~~ lead to terminal positions is the position "1". So we label this P, and now proceed with the usual algorithm:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	...
X	P	N	P	N	N	N	N	P	N	P	N	N	N	N	P	N	P	...

Observe the repeating pattern "PNPNNNN" of length 7. A position's label depends only on the previous 6 labels, since we have seen a pattern longer than this, the pattern will continue, starting again at positions which are 1 greater than multiples of 7.

THIS IS NOT NEEDED, IT WAS ADDED FOR CLARITY.

3. Firstly, we check some small cases in order to try to identify a pattern.

0	1	2	3	4	5	6	7	8	9	10	11	12	...
P	N	N	P	N	N	P	N	N	P	N	N	P	...

It seems that the P-positions are the multiples of 3. Let's prove this. Let $P = \{0, 3, 6, \dots\}$, (ie. the multiples of 3), and let $N = \mathbb{N} \setminus P$. (ie. all other non-negative integers). We need to show:

(1) Terminal positions are in P

(2) From every position in N, there is a move to a position in P

(3) From every position in P, every move is to a position in N.

(1): The terminal position is 0, which is a multiple of 3 $\Rightarrow 0 \in P$.

(2): If $x \in N$, then x is not a multiple of 3. Then either $x-1$ or $x-2$ is a multiple of 3. So either removing 1 coin is a move to P or removing 2 coins is a move to P.

(3): Take any $x \in \mathcal{P}$. Then x is a multiple of 3, ie. $x = 3k$ for some $k \in \mathbb{N}$.
A move from x results in a position $x - 2^t = 3k - 2^t$, for some $t \in \mathbb{N}$.

<u>Claim</u>: $3k - 2^t \in \mathcal{N}$

Pf.

Otherwise, $3k - 2^t \in \mathcal{P} \Rightarrow 3k - 2^t$ is a multiple of 3. $\Rightarrow 3k - 2^t = 3l$, for some $l \in \mathbb{N}$

$\Rightarrow 3(k - l) = 2^t$

$\Rightarrow 2^t$ is a multiple of 3.

But powers of 2 are never multiples of 3.
So $3k - 2^t \in \mathcal{N}$. ✓

So all moves are to $\mathcal{N}$, as required.

4.

14	$=$	1110_2
9	$=$	1001_2
7	$=$	0111_2
5	$=$	0101_2
2	$=$	0010
$\oplus$	$=$	0111

The nim-sum is not equal to 0, so by Bouton's theorem this is an N-position.

The leftmost nonzero column is the second column.

A winning move involves taking any row with a "1" in this column, changing it to "0", and then modifying the digits to the right so that the Nim-sum is 0.
In this case, there are three rows to choose from: 14, 7, 5.

From 14: winning move is to move to $1001_2 = 9$
(i.e. Remove 5 chips from the "14" pile).

From 7: winning move is to move to $0000_2 = 0$
(i.e. Remove the whole "7" pile).

From 5: winning move is to move to $0010_2 = 2$
(i.e. Remove 3 chips from the "5" pile).

5. Positions in "Cheating Nim" are given by lists of the form $(x_1, x_2, \ldots, x_k, s)$, where x_i is the number of coins in pile i on the table, and s is the number of chips in the spare pile under the table. We need to show the P-positions are exactly the positions where

$$x_1 \oplus x_2 \oplus \cdots \oplus x_k = 0.$$

Let $P = \{(x_1, x_2, \ldots, x_k, s) : x_1 \oplus x_2 \oplus \cdots \oplus x_k = 0\}$, and
$N = \{(x_1, x_2, \ldots, x_k, s) : x_1 \oplus x_2 \oplus \cdots \oplus x_k \neq 0\}$.

It is enough to show 3 things:

(1) Terminal positions are in P.

(2) From every position in N, there is a move to a position in P.

(3) From every position in P, every move is to a position in N.

(1) The terminal positions are those where there are no chips left, above or under the table, i.e. $(0, 0, \ldots, 0, 0)$. Since $0 \oplus 0 \oplus \cdots \oplus 0 = 0$, this position is in P.

Same proof → as for usual Nim.

(2) Take any position $(x_1, x_2, \ldots, x_k, s)$ in N. So

$$x_1 \oplus x_2 \oplus \cdots \oplus x_k \neq 0.$$

Writing this nim-sum in column form, choose the left-most column whose nim-sum is 1. Let x_i be any row with a 1 in that column. We can choose x_i' to be such that $x_1 \oplus x_2 \oplus \cdots \oplus x_{i-1} \oplus x_i' \oplus x_{i+1} \oplus \cdots \oplus x_k = 0$, and moreover $x_i' < x_i$, since the left-most column that differs between x_i and x_i' has a "1" in x_i and a "0" in x_i'.

This new position is in P.

Very similar → to proof for usual Nim.

(3) From any position $(x_1, x_2, \ldots, x_k, s)$ in P, we have

$$x_1 \oplus x_2 \oplus \cdots \oplus x_k = 0.$$

A move in this game involves changing exactly one x_i (this is true for our new "cheating" move where we increase the size of one pile, and for the usual Nim move where we decrease the size of one pile). Say that it is the pile x_1 that is changed, to a new integer x_1'. Then, by the cancellation property,

$$x_1' \oplus x_2 \oplus \cdots \oplus x_k \neq x_1 \oplus x_2 \oplus \cdots \oplus x_k = 0.$$

So every move yields a position in $\bar{P}$.

This is question 7(a) in Chapter 1, ~~but~~ broken into several simpler questions.

The last parts are tricky!

6. (a) In position $(50,4)$, there are 50 chips on the table and we can remove 1, 2, 3 or 4 chips. If we remove x chips, the next player can remove at most x chips.

The available moves are: $\underset{\text{TAKE FOUR CHIPS}}{(46,4)}$, $\underset{\text{TAKE 3 CHIPS}}{(47,3)}$, $\underset{\text{TAKE 2 CHIPS}}{(48,2)}$, $\underset{\text{TAKE 1 CHIP}}{(49,1)}$

(b) From a position of the form $(n,1)$, there is exactly one move: to $(n-1,1)$, excluding the terminal position $(0,1)$ from which there are no moves. This is exactly the subtraction game where every player removes exactly one coin.

(0,1)	(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	...
P	N	P	N	P	N	

The label of each position depends only on the previous label, and since we have a repeating pattern "PNPN" at position $(0,1)$, this repeating pattern will continue, with every position $(x,1)$ being a P-position for even x and an N-position for odd x.

(c) For odd x, $(x,2)$ is an N-position since there is a move to $(x-1,1)$ which is a P-position by (b). So it remains to identify $(x,2)$ for even x. There are two moves from $(x,2)$: to $(x-1,1)$ and $(x-2,2)$

using the def. of N/P-positions here

For even numbers x, the position $(x-1,1)$ is an N-position. So $(x,2)$ is an N-position if $(x-2,2)$ is a P-position and $(x,2)$ is a P-position if $(x-2,2)$ is an N-position.

$(0,2)$ is a P-position, so $(2,2)$ is an N-position, and $(4,2)$ is a P-position, ...

In general, $(x,2) \Rightarrow \begin{cases} \text{N-position} & \text{if } x \text{ is not divisible by } 4 \\ \text{P-position} & \text{if } x \text{ is divisible by } 4 \end{cases}$

(d) We will argue that $(40,4)$ is a P-position, and since this is an available move from $(44,43)$, this implies that $(44,43)$ is an N-position, and that an optimal move is $\boxed{\text{to remove 4 chips.}}$

From $(40,4)$, there are 4 available moves:

1. $(39,1)$
2. $(38,2)$
3. $(37,3)$
4. $(36,4)$

From previous parts of the question, we know that $(39,1)$ and $(38,2)$ are N-positions.
$(37,3)$ is also an N-position, since from this position one can move to $(36,1)$, a P-position.

It follows that $(40,4)$ is a P-position if $(36,4)$ is an N-position (and otherwise, $(40,4)$ is an N-position).

By the exact same logic as above, $(36,4)$ is an N-position if $(32,4)$ is a P-position, and an N-position otherwise.

It follows that when x is a multiple of 4, $(x,4)$ alternates between being an N-position and a P-position. Since $(0,4)$ is a P-position, we have that $(4,4)$ is an N-position, $(8,4)$ is a P-position, ... In particular, $(36,4)$ is an N-position and $(40,4)$ is a P-position. So we are done.

(e) To summarize what we have learned in (a)–(d):

$$(x,1) \longrightarrow \begin{cases} N & \text{if } x \text{ is } \underline{\text{not}} \text{ divisible by } 2 \\ P & \text{if } x \text{ is divisible by } 2 \end{cases}$$

$$(x,2) \longrightarrow \begin{cases} N & \text{if } x \text{ is } \underline{\text{not}} \text{ divisible by } 4 \\ P & \text{if } x \text{ is divisible by } 4 \end{cases}$$

$$(x,3) \longrightarrow \begin{cases} N & \text{if } x \text{ is } \underline{\text{not}} \text{ divisible by } 4 \\ P & \text{if } x \text{ is divisible by } 4 \end{cases}$$

$$(x,4) \longrightarrow \begin{cases} N & \text{if } x \text{ is } \underline{\text{not}} \text{ divisible by } 8 \\ P & \text{if } x \text{ is divisible by } 8. \end{cases}$$

For some number m, let 2^k be the next power of 2 larger than m, $2^k > m$. We will argue that:

$$(*) \quad \boxed{(n,m) = \begin{cases} N & \text{if } n \text{ is } \underline{\text{not}} \text{ divisible by } 2^k \\ P & \text{if } n \text{ is divisible by } 2^k. \end{cases}}$$

By choice of 2^k, we have $m \geq 2^{k-1}$. If n is not divisible by 2^k, then n is not divisible by 2^{k-1}. We can remove fewer than m chips so that a multiple of 2^{k-1} chips remains. This is a P-position by $(*)$. Hence (n,m) is an N-position.

It remains to deal with the case that n is divisible by 2^k. In this case, removing ~~fewer~~ any number of chips except 2^{k-1} will result in an N-position, by $(*)$. So, like before, (n,m) is a P-position if

$(n-2^{k-1}, 2^{k-1})$ $\cancel{(n-2^{k-1}, 2^{k-1})}$ is an N-position, and (n,m) is an N-position if $(n-2^{k-1}, 2^{k-1})$ is a P-position.

Since $(0, 2^{k-1})$ is a P-position, $(2^{k-1}, 2^{k-1})$ is an N-position, and $(2 \cdot 2^{k-1}, 2^{k-1})$ is a P-position, and $(3 \cdot 2^{k-1}, 2^{k-1})$ is an N-position, ...

In particular, $(n, 2^{k-1})$ is a P-position exactly when n is a multiple of $2 \cdot 2^{k-1} = 2^k$.

So (n,m) is a P-position exactly when n is divisible by 2^k.