

- (4) This game is equivalent to a game with piles of coins, where each pile of coins corresponds to a group of consecutive ‘H’s. A move consists of removing one or two coins from a single pile, and then (if the player wishes) dividing the pile into two piles (since turning a ‘H’ into a ‘T’ in the middle of a group of consecutive ‘H’s breaks the group into two consecutive lists of ‘H’s).

The position in the question corresponds to $(11, 4)$: two piles, with 11 and 4 coins. Since each move of this game acts on a single pile, a game with multiple piles is the same as the sum-of-games where each term is a single-pile game. So $(11, 4) = (11) + (4)$, as games.

We compute the Sprague–Grundy function for a single pile version of this game.

$$\begin{aligned}
g(0) &= 0 \\
g(1) &= \text{mex} \{g(0)\} = 1 \\
g(2) &= \text{mex} \{g(0), g(1)\} = 2 \\
g(3) &= \text{mex} \{g(1), g(2), g(1, 1)\} = \text{mex} \{g(1), g(2), g(1) \oplus g(1)\} = 3 \\
g(4) &= \text{mex} \{g(2), g(3), g(1) \oplus g(2), g(1) \oplus g(1)\} = 1 \\
g(5) &= \text{mex} \{g(3), g(4), g(1) \oplus g(3), g(2) \oplus g(2), g(1) \oplus g(2)\} = 4 \\
g(6) &= \text{mex} \{g(4), g(5), g(1) \oplus g(4), g(2) \oplus g(3), g(1) \oplus g(3), g(2) \oplus g(2)\} = 3 \\
g(7) &= \text{mex} \{g(5), g(6), g(1) \oplus g(5), g(2) \oplus g(4), g(3) \oplus g(3), g(1) \oplus g(4), g(2) \oplus g(3)\} = 2 \\
g(8) &= \text{mex} \{g(6), g(7), g(1) \oplus g(6), \dots, g(3) \oplus g(4), g(1) \oplus g(5), \dots, g(3) \oplus g(3)\} = 1 \\
g(9) &= \text{mex} \{g(7), g(8), g(1) \oplus g(7), \dots, g(4) \oplus g(4), g(1) \oplus g(6), \dots, g(3) \oplus g(4)\} = 4 \\
g(10) &= \text{mex} \{g(8), g(9), g(1) \oplus g(8), \dots, g(4) \oplus g(5), g(1) \oplus g(7), \dots, g(4) \oplus g(4)\} = 2 \\
g(11) &= \text{mex} \{g(9), g(10), g(1) \oplus g(9), \dots, g(5) \oplus g(5), g(1) \oplus g(8), \dots, g(4) \oplus g(5)\} = 6
\end{aligned}$$

So the Sprague–Grundy value of $(11, 4)$ is $6 \oplus 1 = 7$. To move to a P-position, we need to make a move in the (11) pile so that the new SG value is 1. There are two such moves: turn the (11) pile into $(3, 7)$ or $(7, 3)$. In the original game, this corresponds to the moves:

1. “HHHTHHHHHHHTHHHH”
2. “HHHHHHHTHHHTHHHH”

(Remark. This is a variant of the “Google” question from lectures, and this game is in the book on page “I-24”. For the SG value computation, after the first few values it is not necessary to list all followers, and these values can be computed relatively quickly from the previous values.)