In this note, we will study the sum of a subtraction game and Nim. Let G_1 be the subtraction game where, on a player's turn, they may remove 1 or 2 coins. There are 7 chips initially Let G_2 be the Nim game with four piles, with 1, 3, 3 and 4 chips. We will find a winning move (ie. a move to a P-position) in $G_1 + G_2$.

Recall the Sprague–Grundy Theorem,

Theorem 1. *Let $G_1, G_2, \cdots, G_k$ be games with Sprague–Grundy functions $g_1, g_2, \cdots, g_k$ respectively. Then the Sprague–Grundy function g of the sum game $G_1 + G_2 + \cdots + G_k$ is*

$$g(x_1, x_2, \cdots, x_k) = g_1(x_1) \oplus g_2(x_2) \oplus \cdots \oplus g_k(x_k)$$

Proposition 1. *For Nim$(x_1, x_2, \cdots, x_k)$, the Sprague–Grundy function g is given by*

$$g(x_1, x_2, \cdots, x_k) = x_1 \oplus x_2 \oplus \cdots \oplus x_k$$

Proof. First, let us use this to compute the Sprague–Grundy function for Nim. In the lecture, we saw that it follows immediately from the definition of the Sprague–Grundy function and the rules of Nim that for a *single-pile* game of Nim, we have

$$g_{\text{nim}}(x) = x$$

Now, consider Nim with multiple piles. On a player's turn, they choose one of the piles and then remove some number of chips from that pile. This is equivalent to playing the sum of games, where each game is a single-pile game of Nim. It follows from the Sprague–Grundy theorem that the Sprague–Grundy function for Nim is given by:

$$g(x_1, x_2, \cdots, x_k) = g(x_1) \oplus g(x_2) \oplus \cdots \oplus g(x_k) = x_1 \oplus x_2 \oplus \cdots \oplus x_k$$

□

Next we compute the Sprague–Grundy function for the subtraction game where each player can subtract 1 or 2 coins. The terminal position is 0, so we have $g(0) = 0$. Using the definition, we can check that $g(1) = 1$, $g(2) = 2$, $g(3) = 0$, and so on. The pattern appears to be $0, 1, 2$ repeating, with $g(x) = 0$ whenever x is a multiple of 3. This can be justified in the usual way, this is left as an exercise (see homework 2 question 1, for example).

Finally, we are ready to study $G_1 + G_2$, as defined in the beginning of this document. Let g_1, g_2 be the Sprague–Grundy functions for the subtraction game and Nim respectively. Let g be the Sprague–Grundy function for $G_1 + G_2$. The position we are interested in is the position $(7, (1, 3, 3, 4))$.

$$g(7, (1, 3, 3, 4)) = g_1(7) \oplus g_2(1, 3, 3, 4)$$

by the Sprague–Grundy theorem. We know that $g_1(7) = 1$, since 7 is one more than a multple of 3. We know that

$$g_2(1, 3, 3, 4) = 1 \oplus 3 \oplus 3 \oplus 4 = 5.$$

So

$$g(7, (1, 3, 3, 4)) = 1 \oplus 5 = 4.$$

A winning move is any move that results in a Sprague–Grundy value of 0. Well, we can choose to move in the Nim game and remove the pile with 4 coins. The resulting Sprague–Grundy value is

$$g(7, (1, 3, 3, 0)) = g_1(7) \oplus g_2(1, 3, 3, 0) = 1 \oplus 1 \oplus 3 \oplus 3 = 0$$

So this move results in a P-position, as desired.