Instructions

(1) Write your Name, PID, Section, and Exam Version on the front of your Blue Book.
(2) No calculators or other electronic devices are allowed during this exam.
(3) You may use one page of notes, either written by hand or print on an A4 paper using normal font as in the textbook, but no books or other assistance during this exam.
(4) Write your solutions clearly in your Blue Book.
   (a) Carefully indicate the number and letter of each question and question part.
   (b) Present your answers in the same order as they appear in the exam.
(5) Show all of your work. No credit will be given for unsupported answers, even if correct.
(6) Write your Name and PID and section number on this exam sheet and return it inside the front cover of your Blue Book.

0. (1 point) Carefully read and complete the instructions at the top of this exam sheet and any additional instructions written on the chalkboard during the exam.

1. (6 points) Let \( f(x, y) = x \ln(y) + 2x^2 \).
   a) Compute the differential of \( f(x, y) \).
   b) Find the tangent plane to the surface \( z = f(x, y) \) at the point \((x, y) = (1, 1)\).

2. (6 points) Let \( f(x, y) = xy^2 + x^2 \) and \( g(x, y) = -x + y^2 - 8 \).
   a) Compute the gradient of \( f \) at \((a, b)\). Leave your answer in terms of \((a, b)\).
   b) Compute the gradient of \( g \) at \((a, b)\). Leave your answer in terms of \((a, b)\).
   c) Find all points \((a, b)\) with \( ab \neq 0 \) satisfying i) \( g(a, b) = 0 \) and ii) the gradient of \( f \) at \((a, b)\) and the gradient of \( g \) at \((a, b)\) are parallel.

3. (6 points) Suppose \( z = (x - y)e^x, x = u^2 - v^2, y = u^2 + v^2 \).
   a) Compute \( \frac{\partial z}{\partial u} \). Write your answer in terms of \( u, v \).
   b) Compute \( \frac{\partial z}{\partial v} \). Write your answer in terms of \( u, v \).

4. (6 points) Let \( f(x, y) = xy^2 + 2x^2 - 4xy + 8 \).
   a) Compute the critical points of \( f(x, y) \).
   b) Classify the critical points as local maxima, local minimum, saddle points, or none of these.
Answer

1. (6 points) Let \( f(x, y) = x \ln(y) + 2x^2 \).
   a) Compute the differential of \( f(x, y) \).
   b) Find the tangent plane to the surface \( z = f(x, y) \) at the point \((1, 1)\).
   Sol: a) \( df = (\ln(y) + 4x)dx + (x/y)dy \).
   b) \( f(1, 1) = 2, \ df(1, 1) = 4dx + dy \), so the equation of the desired tangent plane is \( z = 2 + 4(x - 1) + (y - 1) \), or simplify to \( z = 4x + y - 3 \).

2. (6 points) Let \( f(x, y) = xy^2 + x^2 \) and \( g(x, y) = -x + y^2 - 8 \).
   a) Compute the gradient of \( f \) at \((a, b)\). Leave your answer in terms of \((a, b)\).
   b) Compute the gradient of \( g \) at \((a, b)\). Leave your answer in terms of \((a, b)\).
   c) Find all points \((a, b)\) with \( ab \neq 0 \) satisfying i) \( g(a, b) = 0 \) and ii) the gradient of \( f \) at \((a, b)\) and the gradient of \( g \) at \((a, b)\) are parallel.
   Sol: a) \( \nabla f = (b^2 + 2a)\vec{i} + 2ab\vec{j} \), 
   b) \( \nabla g = -\vec{i} + 2y\vec{j} \), 
   c) 
   \[
   \begin{aligned}
   b^2 + 2a &= -\lambda \\
   2ab &= 2b\lambda \\
   -a + b^2 &= 8
   \end{aligned} \quad \Leftrightarrow \quad 
   \begin{aligned}
   a &= \lambda \\
   b^2 &= -3a \\
   -4a - 8 &= 0
   \end{aligned} \quad \Rightarrow \quad 
   \begin{aligned}
   a &= -2 \\
   b &= \pm \sqrt{6}
   \end{aligned}
   
   So the desired points \((a, b)\) are given by \((-2, \pm \sqrt{6})\).

3. (6 points) Suppose \( z = (x - y)e^x, \ x = u^2 - v^2, \ y = u^2 + v^2 \).
   a) Compute \( \frac{\partial z}{\partial a} \). Write your answer in terms of \( u, v \).
   b) Compute \( \frac{\partial z}{\partial v} \). Write your answer in terms of \( u, v \).
   Sol: a) 
   \[
   \frac{\partial z}{\partial u} = (e^x + (x-y)e^x) \cdot 2u + (-e^x) \cdot 2u = ((u^2-v^2)+(u^2-v^2)-(u^2+v^2)) \cdot 2u + (-u^2-v^2) \cdot 2u.
   \]
   b) 
   \[
   \frac{\partial z}{\partial v} = (e^x + (x-y)e^x) \cdot (-2v) + (-e^x) \cdot 2v = ((u^2-v^2)+(u^2-v^2)-(u^2+v^2)) \cdot (-2v) + (-u^2-v^2) \cdot 2v.
   \]

4. (6 points) Let \( f(x, y) = xy^2 + 2x^2 - 4xy + 8 \).
   a) Compute the critical points of \( f(x, y) \).
   b) Classify the critical points as local maxima, local minimum, saddle points, or none of these.
   Sol: 
   \[
   \begin{align*}
   f_x &= y^2 + 4x - 4y = 0 \\
   f_y &= 2xy - 4x = 2x(y - 2) = 0
   \end{align*}
   \Rightarrow \begin{cases} \quad x = 0 & \text{or} \quad y = 2 \\
   y^2 - 4y = 0 & \quad \text{4 + 4x - 8 = 0}
   \end{cases}
   \]
   Thus the critical points are \((0, 0), (0, 4), (1, 2)\).
Check for each critical points:

$D(0,0) = 4 \cdot 0 - (-4)^2 < 0$, so $(0,0)$ is a saddle point.

$D(0,4) = 4 \cdot 0 - (4)^2 < 0$, so $(0,4)$ is a saddle point.

$D(1,2) = 4 \cdot 2 - 0^2 < 0$, and $f_{xx} = 4 > 0$, so $(1,2)$ is a local minimum.