

# Math 20F - Linear Algebra

## Midterm Examination #2 – ANSWERS

Spring, May 23, 2003 — Instructor: Sam Buss

Write your name or initials on every page before beginning the exam.

You have 50 minutes. There are 7 problems. You may not use notes, textbooks, calculators, or other materials during this exam. You must show your work in order to get credit. Good luck!

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Name:

2

1.

a. State the definition of “the range of  $A$ ,  $R(A)$ ”.

ANSWER:  $\{\mathbf{b} : A\mathbf{x} = \mathbf{b} \text{ is consistent}\}$

b. State the definition of “ $\mathbf{u}_1, \dots, \mathbf{u}_k$  is a basis for  $U$ ”.

ANSWER:  $\mathbf{u}_1, \dots, \mathbf{u}_k$  are linearly independent and span  $U$ .

c. State the definition of “ $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is a linear transformation”.

ANSWER: For all  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$  and all  $\alpha, \beta \in \mathbb{R}$ ,

$$f(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha f(\mathbf{x}) + \beta f(\mathbf{y}).$$

d. State the definition of “ $A$  represents the linear transformation  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ ”.

ANSWER: For all  $\mathbf{x} \in \mathbb{R}^n$ ,  $A\mathbf{x} = f(\mathbf{x})$ .

2. Suppose that  $U$  is a subspace of  $\mathbb{R}^n$  and  $\mathbf{u}_1, \dots, \mathbf{u}_k$  is a spanning set for  $U$ .

For each of the following items, tell whether it *MUST* be true. (Write “Yes” if it must be true; write “No” if it is not necessarily true.)

a.  $k \leq n$ . *NO*

b.  $k \geq n$ . *NO*

c. The dimension of  $U$  is at most  $k$ . *YES*

d. The dimension of  $U$  is at least  $k$ . *NO*

e. The dimension of  $U^\perp$  is at least  $k$ . *NO*

f. The dimension of  $U^\perp$  is at least  $n - k$ . *YES*

Name:

3

3. Let  $\mathbf{u}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$ ,  $\mathbf{u}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$ ,  $\mathbf{u}_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$ ,  $\mathbf{u}_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$ ,  $\mathbf{u}_5 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$ .

Let  $U = \text{Span}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5)$ .

- a. Are  $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5$  linearly independent?
- b. Find a subset of  $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5\}$  that is a basis for  $U$ .

ANSWER: a. The five vectors are not linearly independent. One way to know this is that they lie in a 4-dimensional space.

b. Suggested method for solving this is to use row operations on the matrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 \end{pmatrix}$$

This will show that  $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_4$  are a basis for the column space of the matrix, and hence a basis for  $U$ .

(Other answers are possible, but require justification.)

Name:

4

4. Let  $A = \begin{pmatrix} 1 & 2 & 2 & 1 & 1 \\ 2 & 1 & 0 & 1 & 2 \\ 1 & -1 & -2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$ .

- a. Calculate the rank of  $A$ . ANSWER: 3
- b. Calculate the dimension of  $R(A)$ . ANSWER: 3
- c. Calculate the dimension of  $N(A)$ . ANSWER: 2
- d. Calculate the dimension of  $R(A^T)$ . ANSWER: 3
- e. Calculate the dimension of  $N(A^T)$ . ANSWER: 1
- f. Find a basis for the row space of  $A$ . Explain how you can be sure it is a basis. ANSWER:  $(1, 2, 2, 1, 1)^T$ ,  $(0, 1, 1, 0, 0)^T$ , and  $(0, 0, 1, 1, 0)^T$ , is a basis. This is a basis since they are the three non-zero rows in the row echelon form of  $A$ , since the row space of  $A$ 's REF is the same as the row space of  $A$ .

(Other answers are possible, it depends on the order in which you do the row operations.) One possible result for the row echelon form of  $A$  is:

$$\begin{pmatrix} 1 & 2 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

5. Let  $g : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be the linear transformation defined as follows. To calculate  $g(\mathbf{x})$ , you first rotate  $\mathbf{x}$  counterclockwise 45 degrees around the origin, and then you reflect the resulting vector across the  $y$ -axis (the  $x_2$ -axis) to obtain the value of  $g(\mathbf{x})$ . Find a matrix  $A$  that represents  $g$ .

ANSWER:  $A = \begin{pmatrix} -\sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}$ .

Name:

5

6. Find the linear function  $y = f(x)$  that best approximates the data

x	-2	-1	0	2
y	2	1	0	2

in the least squares sense.

ANSWER:  $f(x) = \frac{44}{35} + \frac{1}{35}x$ .

7. Let  $\mathbf{u} = (1, -1, 2)^T$ . The subspace  $U$  of  $\mathbb{R}^3$  is defined by

$$U = \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} \cdot \mathbf{u} = 0\}.$$

- a. Find a basis for  $U^\perp$ .
- b. Find a basis for  $U$ .
- c. Let  $\mathbf{x} = (1, 0, 0)^T$ . Find the vector  $\mathbf{p} \in U$  which is the closest point in  $U$  to  $\mathbf{x}$ .
- d. Let  $g : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be the linear transformation defined by letting  $g(\mathbf{x})$  equal the projection of  $\mathbf{x}$  onto the subspace  $U^\perp$ . Find the matrix that represents the function  $g(\mathbf{x})$ .

ANSWER: The key to this problem is to note that  $U = \text{Span}(\mathbf{u})^\perp$ . Thus,  $U^\perp = \text{Span}(\mathbf{u})$ .

a.  $\mathbf{u}$  (by itself) is a basis for  $U^\perp$ .

b.  $U$  is the nullspace of the  $3 \times 1$  matrix  $(1, -1, 2)$ . Since this is already in rref form, it can be solved immediately. Then, a basis for  $U$  is  $(-2, 0, 1)^T, (1, 1, 0)^T$ .

c. First project  $\mathbf{x}$  onto  $U^\perp$ , to get

$$\mathbf{q} = \mathbf{x} - \frac{(\mathbf{x} \cdot \mathbf{u})\mathbf{u}}{\|\mathbf{u}\|^2} = (1/6, -1/6, 1/3)^T.$$

Then, the answer is  $\mathbf{p} = \mathbf{x} - \mathbf{q} = (5/6, 1/6, -1/3)^T$ .

d. The matrix is

$$\frac{\mathbf{u}\mathbf{u}^T}{\mathbf{u}^T\mathbf{u}} = \frac{1}{6} \begin{pmatrix} 1 & -1 & 2 \\ -1 & 1 & -2 \\ 2 & -2 & 4 \end{pmatrix}$$