Math 20F - Linear Algebra

Final Examination

March 18, 2003

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Write your name or initials on every page before beginning the exam.

You have three hours. There are ten problems. You may use a single $8\frac{1}{2} \times 11$ sheet of notes for reference. (Notes may not be shared!) You may not use calculators or textbooks materials during this exam. Please label your answers clearly. **You must show your work in order to get credit.** Good luck!

Name:

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1. For each of the following, indicate whether it is true or false. To be true, it must always be true.

______ **a.** If A is a square matrix and the columns are an orthonormal set of vectors, then the rows of A are an orthonormal set of vectors.

______ **b.** A change-of-basis matrix (also called a transition matrix) must have an orthogonal set of columns.

______ **c.** If A is a square matrix and represents a linear transformation, then A is non-singular.

______ **d.** If A is a non-singular $n \times n$ matrix and has n distinct eigenvalues, then A has a set of n orthogonal eigenvectors.

2. Let A be the matrix $A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$.

a. What is the rank of A?

b. Find a basis for the null space of A.

3. Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation defined by

$$f : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} x_1 + x_2 \\ 2x_1 - 3x_2 \\ 0 \end{pmatrix}.$$

What matrix represents the transformation f?

4. Let λ be a scalar and let A be the matrix

$$\begin{pmatrix} \lambda & 1 & -1 \\ 1 & 2 & -2 \\ -1 & 1 & 0 \end{pmatrix}.$$

Find a formula for the value of the determinant of A.

5. The following four vectors are linearly dependent.

$$\begin{aligned} \mathbf{u}_1 &= (1\ 1\ 2\ 2)^T \\ \mathbf{u}_2 &= (1\ 2\ 3\ 2)^T \\ \mathbf{u}_3 &= (-1\ 1\ 2\ 1)^T \\ \mathbf{u}_4 &= (2\ 2\ 2\ 1)^T \end{aligned}$$

Find a way to express one of the four vectors as a linear combination of the other three. (You should show the linear combination explicitly.)

6. Let A be the matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 3 \end{pmatrix}$.

a. What is the trace of A?

b. What is the characteristic polynomial of A?

c. Find all the eigenvalues of A and find a eigenvector corresponding to each eigenvalue.

7. Let the following table represent measured values of a function:

x	-1	0	1	2
y	3	3	-3	5

Find the best least squares fit by a quadratic function. In other words, find the quadratic function, $f(x) = c_0 + c_1 x + c_2 x^2$, which best approximates these data values in the least squares sense.

8. Let $U = Span\left(\ (1,-1,0,0)^T,\ (0,1,-1,0)^T\ \right)$.

a. Find an orthonormal basis for U.

b. Let $\mathbf{x} = (1,1,1,1)^T$. What is the vector projection of $\mathbf{x}$ onto U?

9. Let $\mathbf{u}_1 = (1,1,0)^T$ and $\mathbf{u}_2 = (0,1,1)^T$ and $\mathbf{u}_3 = (1,0,1)^T$. Let the linear transformation $f : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$f(\mathbf{x}) \ = \ \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix},$$

where $\alpha_1, \alpha_2, \alpha_3$ are the coordinates of $\mathbf{x}$ relative to the basis $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$. Find the matrix representation of the linear transformation f.

10. Let A be a square matrix. Prove that if λ is an eigenvalue of A, then λ^2 is an eigenvalue of the matrix A^2.