ANSWER KEY

Math 20F - Linear Algebra

Midterm Examination #2

March 5, 2003

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Write your name or initials on every page before beginning the exam.

You have 50 minutes. There are seven problems. You may not use calculators, notes, textbooks, or other materials during this exam. You must show your work in order to get credit. Good luck!

Name:

Student ID:

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Total	

1. Write out the definitions of the following phrases (you may use other technical terms from the course without defining them).

(a) "A is a symmetric matrix."
ANSWER: $A = A^T$.

(b) "A is an skew-symmetric matrix."
ANSWER: $A = -A^T$.

(c) "$\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a basis for V."
ANSWER: $\mathbf{v}_1, \ldots, \mathbf{v}_n$ *are linearly independent and span* V.

(d) "U and V are orthogonal ($U \perp V$)."
ANSWER: For every $\mathbf{u} \in U$ *and every* $\mathbf{v} \in V$, $\mathbf{u}^T\mathbf{v} = 0$.

(e) "U is the orthogonal complement of V ($U = V^\perp$)."
ANSWER: $U = \{\mathbf{u} : \textit{for all } \mathbf{v} \in V,\ \mathbf{u}^T\mathbf{v} = 0\}$

2. Suppose A is an $m \times n$ matrix and that A has rank r.

(a) What is the dimension of $R(A)$? *ANSWER:* r

(b) What is the dimension of $N(A)$? *ANSWER:* $n - r$

(c) What is the dimension of $R(A^T)$? *ANSWER:* r

(d) What is the dimension of $N(A^T)$? *ANSWER:* $m - r$

3. Parts (a) and (b) below concern subspaces of $\mathbb{R}^3$.

(a) Does there exist a pair of subspaces U and V of $\mathbb{R}^3$ such that U and V are orthogonal, but V is not the orthogonal complement of U? If so, give an example.

ANSWER: $U = Span(\mathbf{e}_1)$ *and* $V = Span(\mathbf{e}_2)$ *is one example. Explanation:* $V^\perp = Span(\mathbf{e}_1, \mathbf{e}_3) \neq U$.

(b) Does there exist a pair of subspaces U and V of $\mathbb{R}^3$ such that V is the orthogonal complement of U, but U and V are not orthogonal? If so, give an example.

ANSWER: There is no such U *and* V.

4. Let $A = \begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 6 & 4 & 2 & 0 \end{pmatrix}$.

(a) What is the rank of A? *ANSWER: Rank equals 2.*

(b) Find a set of vectors which is a basis for the column space of A. Justify your answer by explaining how you can be sure that it a basis.

ANSWER: Any two columns of A *can be used to span the column space of* A. *Proof: The column space has dimension two, since* $rank(A) = 2$: *thus any two linearly independent vectors in the column space of* A *span* A. *But, by inspection, no two of the columns of* A *are linearly dependent, since no column is a scalar multiple of another column.*

5. Let $U = Span\left(\ (0,0,1,1)^T,\ (1,0,0,0)^T\ \right)$. Let $\mathbf{x} = (2,2,2,0)^T$. Express $\mathbf{x}$ in the form $\mathbf{x} = \mathbf{p} + \mathbf{q}$ where $\mathbf{p} \in U$ and $\mathbf{q} \in U^\perp$. What are the vectors $\mathbf{p}$ and $\mathbf{q}$?

ANSWER: By inspection, the two vectors that span U are orthogonal; however, the first one is not a unit vector. To get two orthonomal vectors that span U, we convert the first vector to a unit vector:

$$\mathbf{u}_1 = (0,0,1/\sqrt{2},1/\sqrt{2})^T \quad \textit{and} \quad \mathbf{u}_2 = (1,0,0,0)^T.$$

Now, $\mathbf{p} = \langle \mathbf{u}_1, \mathbf{x}\rangle \mathbf{u}_1 + \langle \mathbf{u}_2, \mathbf{x}\rangle \mathbf{u}_2$. *We compute:* $\langle \mathbf{u}_1, \mathbf{x}\rangle = \sqrt{2}$ *and* $\langle \mathbf{u}_2, \mathbf{x}\rangle = 2$. *Thus,* $\mathbf{p} = \sqrt{2}\mathbf{u}_1 + 2\vec{\mathbf{u}}_2 = (2,0,1,1)^T$.

To finish up, $\mathbf{q} = \mathbf{x} - \mathbf{p} = (0,2,1,-1)^T$.

6. Let $\mathbf{u} \in \mathbb{R}^2$ be the unit vector $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)^T$. Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$f(\mathbf{x}) \;=\; \text{the vector projection of } \mathbf{x} \text{ onto } \mathbf{u}.$$

(a) What is the value of $f(\mathbf{e}_1)$?

ANSWER: $(1/4, -\sqrt{3}/4)^T$.

(b) Give the matrix that represents f.

ANSWER: We also find that $f(\mathbf{e}_2) = (-\sqrt{3}/4, 3/4)^T$. *Therefore the matrix that represents f is*

$$\begin{pmatrix} 1/4 & -\sqrt{3}/4 \\ -\sqrt{3}/4 & 3/4 \end{pmatrix}.$$

7. Let the following table represent measured values of a function:

x	0	1	3
y	4	1	2

Find the best least squares fit by a linear function. In other words, find the linear function, $f(x) = c_0 + c_1 x$, which best approximates these data values in the least squares sense.

ANSWER: Let A be the matrix $\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \end{pmatrix}$.

Then

$$A^T A = \begin{pmatrix} 3 & 4 \\ 4 & 10 \end{pmatrix} \quad \text{and} \quad A^T \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ 7 \end{pmatrix}.$$

We need to solve the matrix equation

$$\begin{pmatrix} 3 & 4 \\ 4 & 10 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 7 \\ 7 \end{pmatrix}.$$

When we do this (work omitted), we get $c_0 = 3$ and $c_1 = -1/2$.

Thus, $f(x) = 3 - \frac{1}{2}x$ is the best least squares fit.