

# Math 267a - Propositional Proof Complexity

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### 1 The Unification Problem

#### 1.1 The Problem

Last time we looked at how to solve the system:  $x = \phi, \phi = \psi \rightarrow x$ . In general, given:

- Variables:  $x, y, z, \dots$
- Function Symbols:  $f, g, h, \dots$ , each with specified arity (including constants as 0-ary functions)
- Terms: built from variables and function symbols
- Finite sets of equations:  $s_i \doteq t_i$  (where  $s_i, t_i$  are terms)

the *unification problem* is to find a substitution  $\sigma$  such that  $s_i\sigma = t_i\sigma$  where equality here is equality of symbols.

**Example** Take the system with a single binary function symbol  $f$  and equations:

$$\begin{aligned} x_1 &\doteq f(x_2, x_2) \\ x_2 &\doteq f(x_3, x_3) \\ x_3 &\doteq f(x_4, x_4) \end{aligned}$$

a solution is given by  $\sigma$  such that:

$$\begin{aligned} \sigma(x_3) &= f(x_4, x_4) \\ \sigma(x_2) &= f(f(x_4, x_4), f(x_4, x_4)) \\ \sigma(x_1) &= f(f(f(x_4, x_4), f(x_4, x_4)), f(f(x_4, x_4), f(x_4, x_4))) \end{aligned}$$

As this example shows, in the worst case the solution's size is exponential in the size of the problem (total number of symbols to represent it).

**Example** The system with the single equation  $f(x_1, x_2) = g(x_1, x_2)$  is unsolvable: the function symbols clash.

**Example** The system with equations  $x \doteq g(y)$  and  $y \doteq h(x)$  is unsolvable (the solution must be finite): it yields the derived equation  $x \doteq g(h(x))$  ("occurs check").

We will see that these are the only two kinds of things that can go wrong.

## 1.2 An Algorithm for the Unification Problem

First, notice that terms can be represented by ordered directed acyclic diagrams (ODAGs). By ‘ordered’ we mean that there is a total ordering on each set of edges coming out of a vertex. When represented as ODAGs, the solutions are always polynomial in size.

To solve an unification problem, we first define an equivalence relation  $\approx$  on terms, as follows:

1.  $s \doteq t \rightarrow s \approx t$  (equation in system)
2.  $s = t \rightarrow s \approx t$  (equal as strings)
3.  $f(s_1, s_2, \dots, s_k) \approx f(t_1, t_2, \dots, t_k) \rightarrow \forall i (s_i \approx t_i)$
4.  $s \approx t \rightarrow t \approx s$
5.  $r \approx s \approx t \rightarrow r \approx t$

**Example** Given the system  $f(x, g(y)) \doteq f(h(y), z)$  we have  $x \approx h(y)$  and  $z \approx g(y)$

**Claim** A unification problem is solvable iff (a) There are no  $r \approx s$  so that  $r$  and  $s$  have different principal (outermost) function symbols and (b) The  $\approx$ -equivalence classes are well-ordered (i.e. no cycles) under the extension of the proper subterm relation to the equivalence classes (i.e.,  $[r] \prec [s]$  iff  $r$  is a proper subterm of  $s$ )

Proof:  $[\Rightarrow]$  suppose that  $\sigma$  is a solution. Then it is easy to show that  $r \approx s$  implies  $r\sigma = s\sigma$  holds, by induction on the cases defining  $\approx$ . Similarly, the relation “ $r\sigma$  is a proper subterm of  $s\sigma$ ” is a well-ordering that refines  $r \prec s$ .

$[\Leftarrow]$  Define  $\sigma$  by induction along  $\prec$  as follows. The base elements are of the form  $[x]$ , containing only variables and at most a single constant  $c$ . If  $c \in [x]$  then define  $\sigma(x) = c$ . Otherwise, define  $\sigma(x) = y$  where  $y$  is a new variable that depends only on  $[x]$ .

For the rest, i.e.  $x \in [f(s_1, \dots, s_k)]$  define  $\sigma(x) = (f(s_1, \dots, s_k))\sigma = f(s_1\sigma, \dots, s_k\sigma)$ .

Subclaim:  $v \approx s$  implies  $r\sigma = s\sigma$  (almost immediate).

Partial proof: Suppose  $r = f(r_1, \dots, r_k)$  and  $s = f(s_1, \dots, s_k)$ . Then by the induction hypothesis, we have  $r_i\sigma = s_i\sigma$ .

What we have just described is a polynomial time algorithm. We can obtain the transitive closure using, for example, a breadth-first search or some other similar means (linear in size of graph). In fact, it is quadratic time. Notice that we have considered so far the decision problem; to provide an output in polynomial time we must output ODAGs.

supply reference: Paterson and Wegman provide a linear-time algorithm.

To unify a set of terms  $\{r, s, t, \dots\}$  means to unify the set  $\{r \doteq s, s \doteq t, \dots\}$ . Conversely, to unify  $\{r_i \doteq s_i : 1 \leq i \leq k\}$  is the same as to unify  $\{f(r_1, r_2, \dots, r_k) = f(s_1, s_2, \dots, s_k)\}$

## 2 Extended Frege Systems (Again)

Now we look at an alternative definition of extended Frege systems ( $e\mathcal{F}$ -systems). An  $e\mathcal{F}$ -system is a Frege system ( $\mathcal{F}$ -system) augmented by *the extension rule*. That is, an  $e\mathcal{F}$ -proof consists of formulas  $\phi_1, \dots, \phi_n$  where each  $\phi_i$  is:

- an axiom
- inferred by a rule from previous formulas
- is  $z \leftrightarrow \psi$  where
  - $z$  does not occur in  $\psi$ ,
  - $z$  does not occur in any previous line of the proof, and
  - $z$  does not occur in  $\phi_n$ .

To convert an  $e\mathcal{F}$ -proof into a  $\mathcal{F}$ -proof, proceed as follows:

- Replace  $z$  by  $\phi$  wherever  $z$  occurs
- Replace  $\psi \leftrightarrow \psi$  with a  $\mathcal{F}$ -proof of it ( $O(1)$  lines).

**Theorem 1** (Statman [1]) *An  $n$ -line  $\mathcal{F}$ -proof of  $\phi$  can be converted into an  $e\mathcal{F}$  proof of  $\phi$  with  $O(n + |\phi|)$  lines and  $O(n + |\phi|^2)$  symbols.*

**Theorem 2** *Any two  $e\mathcal{F}$ -systems  $p$ -simulate each other*

**Conjecture 1**  *$\mathcal{F}$ -systems do not simulate  $e\mathcal{F}$ -systems*

Notice that in polynomial-size  $\mathcal{F}$ -proofs, each line is a polynomial-size formula, while in polynomial-size  $e\mathcal{F}$ -proofs, each line is equivalent to a polynomial-size circuit. This converts to circuits; every use of resolution defines a circuit that is then used as input to subsequent circuits:  $\psi(x_1, \dots, x_k, z_1, \dots, z_\ell)$  where  $z_i \leftrightarrow \chi_i(x_1, \dots, x_k, z_i, \dots, z_{i-1})$ .

**Open Problem** Polynomial size formulas have the same expressive power as polynomial circuits

Not only we do not know whether  $\mathcal{F}$  systems simulate  $e\mathcal{F}$  systems and whether  $p$ -size formulas simulate  $p$ -size circuits, we also do not know how to prove either one of these implications if the other one holds.

Proof System	Nonuniform Complexity	Uniform Complexity	Bounded Arithmetic
$\mathcal{F}$	$p$ -size formulas	ALOGTIME	$TNC_1$
$e\mathcal{F}$	$p$ -size circuits	P	$PV/S_2^1$

For references on this table, see Steve Cook's slides from the Edinburgh Complexity Workshop, October 2001, available at

<http://www.cs.toronto.edu/~sacook/edinburgh.ps>

## References

- [1] R. STATMAN, *Complexity of derivations from quantifier-free Horn formulae, mechanical introduction of explicit definitions, and refinement of completeness theorems*, in Logic Colloquium '76, R. Gandy and M. Hyland, eds., Amsterdam, 1977, North-Holland, pp. 505–517.