HW2 MATH170B.

Sec 3.2

#13. Recall Newton's method is 2nd order accuracy.

$\cancel{\|e_n\| < C\|e_{n-1}\|^2}$

$$|e_n| < C|e_{n-1}|^2$$
$$< C'(Ce_{n-2}^2)^2$$
$$< C^{1+2} e_{n-2}^{2^2}$$
$$\cdots$$
$$< C^{2^0+2^1+\cdots+2^{n-1}} e_0^{2^n}$$

where $C \approx \frac{1}{2}\frac{f''(r)}{f'(r)}$, $e_0 = 10^{10}-1$

Then find n, s.t.

$$C^{2^0+2^1+\cdots+2^{n-1}} C_0^{2^n} < 10^{-8}$$

This does not work for this case, because $e_0^{2^n} \to \infty$.

Consider another method,

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$
$$= x_{n-1} - \frac{x_{n-1}^2-1}{2x_{n-1}}$$
$$= \frac{1}{2}x_{n-1} + \frac{1}{2x_{n-1}}$$
$$\le \frac{1}{2}x_{n-1} + \frac{1}{2}$$
$$\le \frac{1}{2}\left(\frac{1}{2}x_{n-2} + \frac{1}{2}\right) + \frac{1}{2}$$
$$\doteq \left(\frac{1}{2}\right)^2 x_{n-2} + \left(\frac{1}{2} + \left(\frac{1}{2}\right)^2\right)$$
$$\le \left(\frac{1}{2}\right)^n x_0 + \left[\left(\frac{1}{2}\right)^1 + \cdots + \left(\frac{1}{2}\right)^n\right]$$
$$\le \left(\frac{1}{2}\right)^n x_0 + 1$$

$\Rightarrow e_n \le \left(\frac{1}{2}\right)^n x_0$

$\Rightarrow$ Find n s.t. $\left(\frac{1}{2}\right)^n x_0 < 10^{-8}$.

#15. Consider

$$0 = f(r) = f(x_n - e_n)$$
$$= f(x_n) - \frac{f'(\xi_n)}{1!} e_n$$

$\Rightarrow f(x_n) = f'(\xi_n) e_n$

$\Rightarrow x_{n+1} = x_n - \frac{f'(\xi_n) e_n}{f'(x_0)}$

$\Rightarrow x_{n+1} - r = x_n - r - \frac{f'(\xi_n) e_n}{f'(x_0)}$

$\Rightarrow |e_{n+1}| = \left|\left(1 - \frac{f'(\xi_n)}{f'(x_0)}\right)e_n\right| \le \max\left\{\left|1 - \frac{\|f'\|_\infty}{f'(x_0)}\right|, \left|1 + \frac{\|f'\|_\infty}{f'(x_0)}\right|\right\} |e_n|$

i.e. $C = \max\left\{ \left|1 - \frac{\|f'\|_u}{f'(x_0)}\right|, \left|1 + \frac{\|f'\|_u}{f'(x_0)}\right| \right\}$,

$s = 1$.

#18. Since $f(x_n) = f'(\xi_n) e_n$

$\Rightarrow x_{n+1} - r = x_n - r - \alpha f'(\xi_n) e_n$

$\Rightarrow |e_{n+1}| = |1 - \alpha f'(\xi_n)| |e_n|$

To make sure this method is linearly convergent, one need $|1 - \alpha f'(\xi_n)| < 1$ for any $\xi_n \in [r, x_n]$.

The sufficient contion on α is

$|1 - \alpha \|f'\|_u| < 1$

and $|1 + \alpha \|f'\|_u| < 1$ $\square$

Sec 3.3

#7. $x_{n+1} = x_n - f(x_n)\left[\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right]$

$$= \frac{x_n f(x_n) - x_n f(x_{n-1}) - x_n f(x_n) + x_{n-1} f(x_n)}{f(x_n) - f(x_{n-1})}$$

$$= \frac{f(x_n) x_{n-1} - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}.$$

The reason why we prefer equ 3 in practice is we can use the result from previous step to save flops.

At step n, let $x_{n+1} - x_n = -f(x_n)\left[\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right] \triangleq h_{n+1}$

At step $n+1$, $x_{n+2} = x_{n+1} - f(x_{n+1})\left[\frac{h_{n+1}}{f(x_{n+1}) - f(x_n)}\right]$.

Sec 3.4

#1. Since s is the fixed point, $F(s)=s$.

and F is contrative

i.e. $|F(x)-F(y)| \leq \lambda |x-y|, \forall x,y \in [a,b]$.

Then $|x_n - s| = |F(x_{n-1}) - F(s)|$

$\leq \lambda |x_{n-1} - s|$

$\cdots$

$\leq \lambda^n |x_0 - s|$, $C = |x_0 - s|$.

The upper bound of C is

$C = |x_0 - s| \leq 2\|F\|_{u[a,b]}$.

#2. By Mean-Value Theorem,

One has $|F(x) - F(y)| = |F'(\xi)| |x-y|$

$\leq \|F'\|_u |x-y| \quad \forall x,y \in [a,b]$.

where $\|F'\|_u = \sup\limits_{\forall x \in [a,b]} |F'(x)| < 1$. (since $F \in C'[a,b]$).

Thus F is contraction mapping.

~~And by Contraction mapping Thm,~~

F may NOT has a fixed-pt.

Since $F: [a,b] \to \mathbb{R}$, pick $F(x) = 100$, and $[a,b] = [0,1]$.

there is no fixed point.

#3. Use Contradiction

Assume there is no fix-~~pient~~ point,

Then Either $F(x)-x>0$ on $[a,b]$ ①

or $F(x)-x<0$ on $[a,b]$. ②

For ①, pick $x=b$, $F(b)-b>0$

$F(b)>b$.

contradiction with $F:[a,b]\to[a,b]$.

For ②, pick $x=a$, $F(a)-a<0$

$F(a)<a$.

contradiction with $F:[a,b]\to[a,b]$.

Therefore, F has a fixed pt.

If $f:\mathbb{R}\to\mathbb{R}$, this statement is not true

Counterexample:

Let $f=x+1$.

then $f(x)-x=1\neq 0$ on $\mathbb{R}$.

$\square$