MATH 170B HW V. and #1 in Mid-term II.

Sec 6.7:

#5. By Thm 2, the derivative keeps the same radius, thus we can take further derivatives term by term.

i.e. $f^{(n)}(x) = \sum_{k=0}^{\infty} \frac{a_k k!}{(k-n)!} (x-c)^{k-n}$, if $|x-c| < r$. $\square$

#16. $\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$, $-\infty < x < \infty$.

then ① $\sin x = \int \cos x \, dx = \sum_{k=0}^{\infty} \int (-1)^k \frac{x^{2k}}{(2k)!} dx$

$= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$

② $\sin x = -\frac{d\cos x}{dx} = -\sum_{k=0}^{\infty} \frac{d \frac{(-1)^k x^{2k}}{(2k)!}}{dx} = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$.

#25. Obviously by Thm 2.

Sec 6.8:

#4 Use Induction.

① $\{P_0\}$ is obviously linear indep.

② Assume $\{P_0, \cdots, P_k\}$ is linear indep.

Then consider $\{P_0, \cdots, P_{k+1}\}$

$\sum_{i=0}^{k+1} a_i P_i = 0$, Since only P_{k+1} has term x^{k+1}, so $a_{k+1} = 0$

$\Rightarrow \sum_{i=0}^{k} a_i P_i = 0 - a_{k+1} P_{k+1} = 0$, by assumption, $a_i = 0, i = 0, \cdots, k$

$\Rightarrow a_i = 0, \cdots, k+1$. $\square$.

#5 Show $\langle f, g \rangle = \sum_{i=1}^{n} \langle f, u_i \rangle \langle g, u_i \rangle$, $[u_1, \cdots, u_n]$ is orthonormal set.

Proof: Since $f = \sum_{i=1}^{n} \alpha_i u_i$, $g = \sum_{i=1}^{n} \beta_i u_i$,

$$\langle f, g \rangle = \langle \sum_{i=1}^{n} \alpha_i u_i, \sum_{j=1}^{n} \beta_j u_j \rangle \quad \text{by orthognality}$$

$$= \sum_{i=1}^{n} \langle \alpha_i u_i, \beta_i u_i \rangle, \quad \text{by } \|u_i\| = 1$$

$$= \sum_{i=1}^{n} \alpha_i \beta_i$$

$$\sum_{i=1}^{n} \langle f, u_i \rangle \langle g, u_i \rangle = \sum_{i=1}^{n} \langle \sum_{j=1}^{n} \alpha_j u_j, u_i \rangle \langle \sum_{j=1}^{n} \beta_j u_j, u_i \rangle$$

$$= \sum_{i=1}^{n} \langle \alpha_i u_i, u_i \rangle \langle \beta_i u_i, u_i \rangle$$

$$= \sum_{i=1}^{n} \alpha_i \beta_i .$$

$\square$

#15 Proof: Let $\{u_1, \cdots, u_n\}$ be a orthogonal set, and all $u_i \neq 0$.

Consider $\sum_{i=1}^{n} a_i u_i = 0$

then WLOG $\langle \sum_{i=1}^{n} a_i u_i, u_i \rangle = \langle 0, u_i \rangle = 0$

$\Rightarrow a_i \langle u_i, u_i \rangle = a_i \|u_i\|^2 = 0$, since $u_i \neq 0$

$\Rightarrow a_i = 0$, $i = 1, \cdots, n$.

Therefore, $\{u_1, \cdots, u_n\}$ is linear indep set.

$\square$

#16 Proof: Let $Af_1 = \lambda_1 f_1$

$Af_2 = \lambda_2 f_2$, $\lambda_1 \neq \lambda_2$

then $\langle f_1, Af_2\rangle = \boxed{\langle Af_1, f_2\rangle = \langle \lambda_1 f_1, f_2\rangle} = \lambda_1 \langle f_1, f_2\rangle$

by self adjoint

$= \langle f_1, \lambda_2 f_2\rangle = \lambda_2 \langle f_1, f_2\rangle$.

Finally, we have $\lambda_2 \langle f_1, f_2\rangle = \lambda_1 \langle f_1, f_2\rangle$, since $\lambda_1 \neq \lambda_2$

$\Rightarrow \langle f_1, f_2\rangle = 0$, i.e $f_1 \perp f_2$.

$\square$.

#18. Proof: Thm 7

① $P_n(\alpha f + \beta g) = \sum_{i=1}^{n} \langle \alpha f + \beta g, u_i\rangle u_i = \alpha \sum_{i=1}^{n} \langle f, u_i\rangle u_i + \beta \sum_{i=1}^{n} \langle g, u_i\rangle u_i$

$= \alpha P_n f + \beta P_n g$, P_n is Linear.

Then show P_n is "onto".

$\forall g \in U_n$, ~~one can~~ i.e. $g = \sum_{i=1}^{n} a_i u_i$ and $g \in E$ also.

then consider $P_n g = \sum_{i=1}^{n} \langle g, u_i\rangle u_i = \sum_{i=1}^{n} \langle \sum_{j=1}^{n} a_j u_j, u_i\rangle u_i$

$= \sum_{i=1}^{n} a_i u_i = g$,

thus P_n is surjective.

② Show $P_n^2 = P_n$

In ①, we already showed if $g \in U_n$, then $P_n g = g$,

thus ~~$P_n(P_n\ P_n g = P_n(P_n g) = P_n g$~~

For $f \in E$, $P_n^2 f = P_n(P_n f) = P_n f$. , i.e. $P_n^2 = P_n$.

③ Let $f \in E$, $P_n f = \sum_{i=1}^{n} \langle f, u_i \rangle u_i \in U_n$, $\forall g \in U_n$, $g = \sum_{i=1}^{n} \beta_i u_i$.

Then
$$\langle f - P_n f, g \rangle = \langle f - \sum_{i=1}^{n} \langle f, u_i \rangle u_i, \sum_{i=1}^{n} \beta_i u_i \rangle$$
$$= \langle f, \sum_{i=1}^{n} \beta_i u_i \rangle - \langle \sum_{i=1}^{n} \langle f, u_i \rangle u_i, \sum_{j=1}^{n} \beta_j u_j \rangle$$
$$= \sum_{i=1}^{n} \langle f, u_i \rangle \beta_i - \sum_{i=1}^{n} \langle f, u_i \rangle \langle u_i, u_i \rangle \beta_i$$

Since $\|u_i\| = 1$ orthonormal.
$$= \sum_{i=1}^{n} \langle f, u_i \rangle \beta_i - \sum_{i=1}^{n} \langle f, u_i \rangle \beta_i \ \cancel{\text{Since}} = 0.$$

Since g in arbitrary, $f - P_n f \perp U_n$.

④ Consider
$$\|f - P_n f\|^2 = \langle f - P_n f, f - P_n f \rangle$$
$$= \langle f, f - P_n f \rangle - \underbrace{\langle P_n f, f - P_n f \rangle}_{0 \text{ by } ③}$$
$$= \langle f - g + g, f - P_n f \rangle, \quad \forall g \in U_n$$
$$= \langle f - g, f - P_n f \rangle + \underbrace{\langle g, f - P_n f \rangle}_{0 \text{ by } ③}$$

by Cauchy-schwarz $\le \|f - g\| \, \|f - P_n f\|$.

$\Rightarrow$ Divide $\|f - P_n f\|$, $\|f - P_n f\| \le \|f - g\|$, $\forall g \in U_n$.

i.e. $P_n f$ is best approximation.

Let $f, g \in E$

⑤ $\langle P_n f, g \rangle =$
$$\langle P_n f, g - P_n g + P_n g \rangle = \underbrace{\langle P_n f, g - P_n g \rangle}_{0 \text{ by } ③} + \langle P_n f, P_n g \rangle$$
$$= \langle P_n f - f + f, P_n g \rangle$$
$$= \underbrace{\langle P_n f - f, P_n g \rangle}_{0 \text{ by } ③} + \langle f, P_n g \rangle$$

D.

#21 By Thm 5: from Example 2,

we have $P_0=1$, $P_1=x$, $P_2=x^2-\frac{1}{3}$.

then $P_3(x)=(x-a_3)P_2-b_3P_1$

where $a_3=\dfrac{\langle xP_2,P_2\rangle}{\langle P_2,P_2\rangle}=\dfrac{\int_{-1}^{1}x(x^2-\frac{1}{3})^2dx}{\int_{-1}^{1}(x^2-\frac{1}{3})^2dx}=0$

$b_3=\dfrac{\langle xP_2,P_1\rangle}{\langle P_1,P_1\rangle}=\dfrac{\int_{-1}^{1}x(x^2-\frac{1}{3})x\,dx}{\int_{-1}^{1}x^2dx}=\dfrac{4}{15}$

$\Rightarrow P_3=xP_2-\frac{4}{15}P_1=x^3-\frac{3}{5}x.$

Use the same algorithm to find P_4 and P_5.

#22. Here the space becomes $C[0,1]$,

and inner-product is $\int_0^1 fg\,dx$.

let $P_0=1$, $P_1=x-a_1$

① Find $a_1=\dfrac{\langle xP_0,P_0\rangle}{\langle P_0,P_0\rangle}=\dfrac{\int_0^1 x\,dx}{\int_0^1 1\,dx}=\dfrac{1}{2}$

$\Rightarrow P_1=x-\frac{1}{2}$

② $P_2=(x-a_2)P_1-b_2P_0$

where $a_2=\dfrac{\langle xP_1,P_1\rangle}{\langle P_1,P_1\rangle}=\dfrac{\int_0^1 x(x-\frac{1}{2})^2dx}{\int_0^1(x-\frac{1}{2})^2dx}=\dfrac{1}{2}$

$b_2=\dfrac{\langle xP_1,P_0\rangle}{\langle P_0,P_0\rangle}=\dfrac{\int_0^1 x(x-\frac{1}{2})dx}{\int_0^1 1\,dx}=\dfrac{1}{12}$

$\Rightarrow P_2=(x-\frac{1}{2})(x-\frac{1}{2})-\frac{1}{12}=x^2-x+\frac{1}{6}$.

Similar for P_3.

□

#1 of Mid-term II.

(a). p is a direction of decrease if $\exists\ \delta>0$ s.t.

$f(x^0+\alpha p) < f(x^0)$ for all $\alpha \in (0,\delta)$ at x^0.

(b). p is a descent direction for f at x^0 if $\nabla f(x_0)^T p < 0$.

(c). $\nabla f(x_0)=0$, then x_0 is a stationary pt.

(d). $\exists$ open ball $B(x^*,r)$ s.t. $f(x^*) \le f(y)$, $\forall y \in B(x^*,r)$,

then x^* is a local minimizer of f.

(e). First order necessary condition, $\nabla f(x^*)=0$

Second order " " , $p^T \nabla^2 f(x^*) p \geq 0$, $\forall p$.

(f) Sufficient condition, $\nabla f(x^*)=0$

and $p^T \nabla^2 f(x^*) p > 0$, $\forall p$.