MATH 171B HW V.

#5.1

(a) ① Substitute $x(\alpha)$ in to $c(x)$

then $c(x(\alpha))=0$, for $\alpha>0$

② $x(0)=(0,0)$, where $(0,0)$ is a feasible point.

③ $\left.\frac{dx(\alpha)}{d\alpha}\right|_{\alpha=0} = (\cos\alpha, -\sin\alpha)\Big|_{\alpha=0}=(1,0)\neq 0$.

All for ① ② and ③, $x(\alpha)$ is a feasible path.

And tangent of $x(\alpha)$ on $(0,0)$ is $\left.\frac{dx(\alpha)}{d\alpha}\right|_{\alpha=0}=(1,0)$.

(b). $f(x(\alpha))=3(\cos\alpha-1)+\sin^2\alpha+(\cos\alpha-1)^2$

and $f(x(0))=0$.

(c). Defn: $L(x,\lambda)=3x_2+x_1^2+x_2^2-\lambda(x_1^2+(x_2+1)^2-1)$

$$J(x)=\begin{pmatrix}2x_1\\ 2(x_2+1)\end{pmatrix}=\nabla c(x)$$

$$\Rightarrow \nabla L(x,\lambda)=\nabla f-\nabla\lambda(c(x))=\begin{pmatrix}(2-2\lambda)x_1\\ (2-2\lambda)x_2+3-2\lambda\\ -x_1^2-(x_2+1)^2+1\end{pmatrix}$$

$$\text{And } \nabla^2_{xx}L(x,\lambda)=\nabla^2_{xx}f-\lambda\nabla^2_{xx}c(x)$$

$$=\begin{pmatrix}2&0\\0&2\end{pmatrix}-\lambda\begin{pmatrix}2&0\\0&2\end{pmatrix}$$

$$=\begin{pmatrix}2-2\lambda&0\\0&2-2\lambda\end{pmatrix}.$$

(d). (i) $(0,0)^T$ is feasible.

(ii) $\exists \lambda = 3/2$ s.t. $\nabla L = 0$

(iii) $\nabla_{xx}^2 L(x,\lambda)\Big|_{\substack{x=0\\ \lambda=3/2}} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

Then $\exists\ p = (1,0)^T \in N(J(x^*))$, s.t.

$$p^T \nabla_{xx}^2 L(x,\lambda) p\Big|_{\substack{x=0\\ \lambda=3/2}} = (-1,0)\begin{pmatrix}1\\0\end{pmatrix} = -1 < 0.$$

Fail the necessary condition.

then $(0,0)^T$ is not a minimizer.

5.2.

(a). KKT condition is 1st order necessary condition.

i.e $\nabla L = 0$.

Defn $L(x,\lambda) = x_1^2 + 2x_2^2 - \lambda(x_1 + x_2 - 1)$

$\Rightarrow$ ~~L~~ $\nabla L = \begin{pmatrix} 2x_1 - \lambda \\ 4x_2 - \lambda \\ x_1 + x_2 - 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{cases} x_1 = \frac{2}{3} \\ x_2 = \frac{1}{3} \\ \lambda = 4/3 \end{cases}$, with $x^* = (\frac{2}{3}, \frac{1}{3})^T$.

(b). Then verify $x^* = (\frac{2}{3}, \frac{1}{3})^T$.

Since $x_1 = 1 - x_2$, then objective function is

$(1-x_2)^2 + 2x_2^2 = 3x_2^2 - 2x_2 + 1 \Rightarrow$ minimizer is $x_2 = \frac{1}{3}$

$\Rightarrow (\frac{2}{3}, \frac{1}{3})^T$ is exactly an optimal point.

(b) Let $L(x,\lambda) = x_1^3 + x_2^3 - \lambda(x_1+x_2-1) = 0$

$$\nabla L = \begin{pmatrix} 3x_1^2-\lambda \\ 3x_2^2-\lambda \\ x_1+x_2-1 \end{pmatrix} = 0 \Rightarrow \begin{cases} x_1 = \frac{1}{2} \\ x_2 = \frac{1}{2} \\ \lambda = 3/4 \end{cases} \Rightarrow x^* = (\tfrac{1}{2}, \tfrac{1}{2})^T.$$

Then verify $(\frac{1}{2}, \frac{1}{2})^T$,

Let $x_1 = 1 - x_2$, then objective function

$$(1-x_2)^3 + x_2^3 = 3x_2^2 - 3x_2 + 1$$

$\Rightarrow$ minimizer $x_2 = \frac{1}{2}$

$\Rightarrow$ $(\frac{1}{2}, \frac{1}{2})^T$ is exactly a optimal point.